

DERIVED GALOIS DEFORMATION RINGS

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ABSTRACT. We define a derived version of Mazur’s Galois deformation ring. It is a pro-simplicial ring \mathcal{R} classifying deformations of a fixed Galois representation to simplicial coefficient rings; its zeroth homotopy group $\pi_0\mathcal{R}$ recovers Mazur’s deformation ring.

We give evidence that these rings \mathcal{R} occur in the wild: For suitable Galois representations, the Langlands program predicts that $\pi_0\mathcal{R}$ should act on the homology of an arithmetic group. We explain how the Taylor–Wiles method can be used to upgrade such an action to a graded action of $\pi_*\mathcal{R}$ on the homology.

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1. INTRODUCTION

1.1. The Langlands program posits a bijection between automorphic forms and Galois representations. The study of p -adic congruences leads to the definition of a “ p -adic moduli space of automorphic forms,” namely the spectrum $\mathrm{Spec}(\mathbf{T})$ of a suitable Hecke algebra, and a “ p -adic moduli space of (geometric, e.g. crystalline) Galois representations,” the spectrum $\mathrm{Spec}(\mathbf{R})$ of Mazur’s Galois deformation ring. Proving that the natural map

$$(1.1) \quad \mathbf{R} \xrightarrow{\sim} \mathbf{T}$$

is an isomorphism is the basis of modern proofs of modularity, after the work of Wiles and Taylor–Wiles [39, 34].

The purpose of this paper is to construct a derived version \mathcal{R} of \mathbf{R} , a pro-finite simplicial ring; it represents Galois deformations with coefficients in simplicial commutative rings. It is unsurprising that a derived version should play a role: speaking informally, the locus of crystalline Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is obtained by intersecting the space of Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the space of local geometric representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and the intersection is not, in general, transverse.

The existence of \mathcal{R} follows by applying the derived Schlessinger criterion of Lurie [22]. The first part of this paper is a leisurely exposition of this criterion and also supplies various basic results that are useful when applying it. The goal of the second part is to explain how \mathcal{R} arises naturally in the Taylor–Wiles method, or rather the obstructed version of this method developed by Calegari and Geraghty: [6, 15, 20].

Our initial motivation for this construction was the numerology of the Betti numbers for an arithmetic group Γ . For instance, if $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$, it appears that the exterior algebra of a vector space of dimension $\delta := \lfloor \frac{n-1}{2} \rfloor$ acts on $H^*(\Gamma, \overline{\mathbb{Q}_p})_\chi$, where the subscript χ means that we take the χ -eigenspace for a tempered character χ of the Hecke algebra, and there is a similar story for any arithmetic group (see [38] for elaboration and discussion). Now let \mathcal{O} be the ring of integers of $\overline{\mathbb{Q}_p}$ and $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_n(\mathcal{O})$ the Galois representation attached to χ and $\overline{\rho}$ its mod p reduction. Suppose the standard conjecture that ρ does not have characteristic zero crystalline deformations. Then $\pi_* \mathcal{R}_{\overline{\rho}} \otimes_{\pi_0 \mathcal{R}_{\overline{\rho}}} \overline{\mathbb{Q}_p}$ is isomorphic to an exterior algebra on δ generators, where the map $\pi_0 \mathcal{R}_{\overline{\rho}} \rightarrow \overline{\mathbb{Q}_p}$ is the one associated to ρ .

This numerical coincidence naturally suggests that $\pi_* \mathcal{R}_{\overline{\rho}}$ might in fact freely act on the integral homology $H_*(\Gamma, \mathcal{O})_{\overline{\rho}}$, where the subscript $\overline{\rho}$ on homology means that we localize at the corresponding ideal of the Hecke algebra. This should generalize the way that the usual deformation ring acts on the homology of a modular curve, and this is indeed what we establish under suitable hypotheses. (Note that the precise hypotheses are rather involved, but we give references to where they are specified; also, in the main text, we work with a specific finite field k and its Witt vectors, rather than $\overline{\mathbb{F}_p}$ and \mathcal{O} as above.)

Part of the main theorem: Assume the existence of Galois representations attached to (possibly torsion) homology classes (Conjecture 6.1); this amounts to giving an action of $\pi_0 \mathcal{R}_{\overline{\rho}}$ on $H_*(\Gamma, \mathcal{O})_{\overline{\rho}}$. Then, under conditions on $\overline{\rho}$ that include enforcing “minimal level,” (§10, §13.1), the homology $H_*(\Gamma, \mathcal{O})_{\overline{\rho}}$ admits a free action of $\pi_* \mathcal{R}_{\overline{\rho}}$, extending the action of $\pi_0 \mathcal{R}_{\overline{\rho}}$.

For more, see §1.4 and the exact result is Theorem 14.1. Our simplifying assumptions on $\overline{\rho}$ are certainly restrictive; nonetheless we expect the statement above to be valid without any local assumptions, at least as long as one deals with crystalline representations in the Fontaine–Laffaille range. See also Remark 1.1.

The proof amounts to using the Taylor–Wiles method to upgrade the action of the usual deformation ring $\pi_0 \mathcal{R}$ on $H_*(\Gamma, \mathcal{O})_{\overline{\rho}}$ to a graded action of $\pi_* \mathcal{R}$. Very roughly speaking, the Taylor–Wiles method shows that one can lift to a situation with unobstructed deformation theory, i.e. where the analogue \mathcal{R}' of \mathcal{R} satisfies $\mathcal{R}' \simeq \pi_0 \mathcal{R}'$; and then one descends. This method of defining the action of $\pi_* \mathcal{R}$ is very indirect, and we hope that it can be replaced by a better, more direct construction.

In more detail: the method of Calegari and Geraghty has already shown that the homology $H_*(\Gamma, \mathcal{O})_{\overline{\rho}}$ has the structure of a free module under a certain Tor-algebra that arises in the Taylor–Wiles limit process. The meaning of this Tor-algebra is *a priori* obscure, because it depends on all the choices made in that limit process. So what we really do is to identify this Tor-algebra with a more intrinsic object, $\pi_* \mathcal{R}$.

What would be even better would be to refine this result by giving a chain level action of the simplicial ring \mathcal{R} on the chain complex of Γ .

After a brief review (§1.2) of simplicial rings, we give a quick overview of the definitions (§1.3) and explain what we prove about it (§1.4).

1.2. Simplicial commutative rings. In this paper we shall write SCR for the category of simplicial commutative rings. Since it is not standard in the number-theory literature, let us briefly recall this concept. This is intended informally, and not as a substitute for a rigorous introduction; for that see [12, 24]. Also, note that we are only ever interested in simplicial *commutative* rings, and occasionally will drop the word “commutative.”

For a topological commutative ring \mathbf{R} , the homotopy groups $\pi_*(\mathbf{R}, 0)$ with basepoint $0 \in \mathbf{R}$ carry the structure of a graded ring, whose addition is defined by pointwise addition of continuous maps into \mathbf{R} . To define the multiplication we represent classes in $\pi_j(\mathbf{R}, 0)$ by maps $[0, 1]^j \rightarrow \mathbf{R}$ sending the boundary to 0, and then we use the maps $[0, 1]^j \times [0, 1]^k \rightarrow [0, 1]^{j+k}$.

One can extract from \mathbf{R} a purely algebraic object – a model example of a simplicial commutative ring – which carries enough information to recover the graded ring $\pi_* \mathbf{R}$: Let $\mathcal{R}_n = \text{Sing}_n(\mathbf{R})$ be the set of continuous maps from the n -simplex Δ^n into \mathbf{R} . The inclusion of the n -simplex as the i th face of the $(n + 1)$ -simplex

gives a map $d_i : \mathcal{R}_{n+1} \rightarrow \mathcal{R}_n$; similarly, collapsing an $(n+1)$ -simplex to its i th face gives maps $s_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n+1}$. These maps $d_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1}$ and $s_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n+1}$, $i = 0, \dots, n$, satisfy various axioms, such as $d_i d_j = d_{j-1} d_i$ when $i < j$.

It is possible to recover the homotopy groups $\pi_* \mathbf{R}$ from the collection of \mathcal{R}_n and the maps d_i and s_i : one builds a CW-complex $|\mathcal{R}|$, the “geometric realization,” by gluing one copy of Δ^n for each element of \mathcal{R}_n , and then there is a canonical map $|\mathcal{R}| \rightarrow \mathbf{R}$ which induces isomorphisms on all homotopy groups.

A simplicial commutative ring is a collection of (commutative) rings \mathcal{R}_n ($n \geq 0$) modelled on this situation: i.e. equipped with maps $d_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1}$ and $s_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n+1}$ satisfying the same axioms as the model case above. To such a simplicial \mathcal{R} we may associate homotopy groups $\pi_p \mathcal{R}$ in a way that will recover $\pi_* \mathbf{R}$ in the model case above (just take π_p of the geometric realization $|\mathcal{R}_n|$). For example, $\pi_0 \mathcal{R}$ is the quotient of \mathcal{R}_0 by the ideal $(d_0 - d_1)(\mathcal{R}_1)$. The direct sum

$$(1.2) \quad \pi_* \mathcal{R} = \bigoplus_p \pi_p(\mathcal{R})$$

inherits the structure of a graded ring.

A map $f : \mathcal{R} \rightarrow \mathcal{R}'$ of simplicial commutative rings is a *weak equivalence* if the induced map of homotopy groups is an isomorphism. This is equivalent to the map of geometric realizations being a homotopy equivalence.

Any (usual, non-simplicial) commutative ring R gives rise to a simplicial ring, in which we take all \mathcal{R}_n ’s to equal R , and all d_i and s_i are the identity map of R . By a slight abuse of notation, the resulting simplicial commutative ring shall often be denoted by the same letter R . Objects of SCR arising in this way are called *constant*, and this construction gives a full and faithful embedding of the category of commutative rings into SCR. This embedding is right adjoint to the functor π_0 from SCR to commutative rings. An object $\mathcal{R} \in \text{SCR}$ is weakly equivalent to a constant object if and only if the natural map $\mathcal{R} \rightarrow \pi_0 \mathcal{R}$ is a weak equivalence, which in turn happens if and only if $\pi_i \mathcal{R} = 0$ for all $i > 0$. Let us say that \mathcal{R} is *homotopy discrete* when that happens.

Finally, for k a field, we will be interested in “Artinian” simplicial rings over k : simplicial commutative rings \mathcal{R} equipped with a homomorphism $\pi_0 \mathcal{R} \rightarrow k$ and with the properties that $\pi_0 \mathcal{R}$ is Artin local in the usual sense, $\pi_i \mathcal{R}$ vanishes for all large enough i , and finally each $\pi_i \mathcal{R}$ is a finitely generated $\pi_0 \mathcal{R}$ -module. We denote by Art_k the category with objects the Artinian simplicial rings over k , and the morphisms are those morphisms of simplicial commutative rings that commute with the map to k .

It shall be important later on that these categories come with *simplicial enrichments*: for objects A and B of SCR the set of morphisms $A \rightarrow B$ is the set of 0-simplices in a simplicial set $\text{Hom}_{\text{SCR}}(A, B)$, and similarly for the category Art_k . In particular any representable functor naturally takes values in simplicial sets.

1.3. Overview of the derived deformation ring. For simplicity in this section, we gloss over several technical points – in particular, the role of categories of pro-objects, and we talk only about GL_n instead of a general algebraic group.

For a finite set of primes S , let

$$\Gamma_S = \pi_1^{\mathrm{et}}(\mathrm{Spec} \mathbb{Z}[1/S]) = \mathrm{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q}),$$

the Galois extension of the largest extension $\mathbb{Q}^{(S)}$ unramified outside S , and we fix k a finite field whose characteristic divides S . We fix a representation

$$\bar{\rho} : \Gamma_S \rightarrow \mathrm{GL}_n(k)$$

which we suppose to have centralizer consisting only of scalar matrices. Then, informally, Mazur’s (non-simplicial) deformation ring R is the completed local ring of the representation variety of the group Γ_S at the representation $\bar{\rho} : \Gamma_S \rightarrow \mathrm{GL}_n(k)$; more precisely R represents the functor $\mathrm{Def}_{\bar{\rho}}$ which sends an Artin ring $A \rightarrow k$ augmented over k to the set

$$\mathrm{Def}_{\bar{\rho}}(A) = (\text{lifts } \tilde{\rho} : \Gamma_S \rightarrow \mathrm{GL}_n(A)) / (\text{kernel of } \mathrm{PGL}_n(A) \rightarrow \mathrm{PGL}_n(k)).$$

Explicitly, then, there is a natural bijection of set-valued functors

$$(1.3) \quad \mathrm{Hom}(R, -) \rightarrow \mathrm{Def}_{\bar{\rho}}(-).$$

(where, on the left hand side, we consider only homomorphisms that commute with the augmentations to k).

The *derived* version \mathcal{R} represents the functor which sends a *simplicial* Artin ring A , augmented over k , to a *simplicial set* $\mathrm{Def}_{\bar{\rho}}(A)$, defined similarly to the above but replacing the set of lifts $\tilde{\rho}$ by a suitable simplicial set. For this, one needs to generalize the objects appearing in the above definition.

To give the reader some sense of the subtleties involved, let us briefly outline one way of carrying out this generalization, although we proceed differently in the text: For a simplicial commutative ring A , we may define $\mathrm{GL}_n(A)$ as the simplicial monoid consisting of those path components of $M_n(A) = A^{n^2}$ which map to $\mathrm{GL}_n(\pi_0 A) \subset M_n(\pi_0 A)$. (There is a more naive definition where one applies GL_n level-wise to A . This would define a functor into simplicial groups but would be no good for our purposes: it would carry some morphisms $A \rightarrow A'$ in SCR which are homotopy equivalences into maps which are not homotopy equivalences.) Next, homomorphisms $\Gamma_S \rightarrow \mathrm{GL}_n(A)$ must be defined in a derived sense. At least if Γ_S is a usual group and not a pro-group, one definition of the space of derived homomorphisms $\Gamma_S \rightarrow \mathrm{GL}_n(A)$ proceeds by choosing a “cofibrant replacement” of Γ_S , which roughly speaking means a simplicial monoid Γ_\bullet equipped with a homotopy equivalence $|\Gamma_\bullet| \rightarrow \Gamma_S$ such that Γ_p is a free associative monoid on a (possibly infinite) set for all $p \geq 0$. Then a zero-simplex of $R\mathrm{Hom}(\Gamma_S, \mathrm{GL}_n(A))$ is a map of simplicial monoids $\Gamma_\bullet \rightarrow \mathrm{GL}_n(A)$. More generally, a p -simplex of $R\mathrm{Hom}(\Gamma_S, \mathrm{GL}_n(A))$ would be a map of simplicial monoids $\Gamma_\bullet \rightarrow \mathrm{GL}_n(A^{\Delta[p]})$, where $A^{\Delta[p]}$ is the mapping space $\mathrm{Hom}(\Delta[p], A)$, which itself has the structure of a simplicial commutative ring. Having said this, in our later presentation, we will use a different approach that works better for general groups (not just GL_n).

That the resulting functor is representable by some (pro-) simplicial ring \mathcal{R} follows from the derived Schlessinger criterion of Lurie. What this means (ignoring the “pro” subtlety for now) is that there is a natural transformation

$$(1.4) \quad \mathrm{Hom}(\mathcal{R}, -) \rightarrow \mathrm{Def}_{\bar{\rho}}(-)$$

of functors valued in simplicial sets, and this natural transformation induces a weak equivalence of simplicial sets for each input $A \rightarrow k$. (Recall that the mapping spaces between two simplicial rings itself has the structure of a simplicial set).

In the usual setting, the bijection (1.3) determines R up to unique isomorphism. This is no longer true *strictly* for (1.4), only up to homotopy: any two \mathcal{R} ’s will be homotopy equivalent as pro-simplicial rings, and in a suitable sense there will be a *contractible* space of comparison maps. Thus we should, strictly speaking, speak only of “a representing ring.” However, the image of \mathcal{R} in a suitable homotopy category is still defined up to a unique isomorphism (see §3.5 for discussion) and therefore associated invariants such as the graded ring $\pi_*\mathcal{R}$ are again determined up to unique isomorphism.

For formal reasons (see Lemma 7.1) $\pi_0\mathcal{R}$ will be isomorphic to Mazur’s deformation ring R , but in fact, we should heuristically expect \mathcal{R} to usually be homotopy discrete. More precisely there are always natural maps

$$(1.5) \quad \mathcal{R} \longrightarrow \pi_0\mathcal{R} \xrightarrow{\cong} R.$$

and we heuristically expect the first map to be a weak equivalence; this is substantially equivalent (see Lemma 7.5) to the folklore conjecture:

Conjecture: The usual (unrestricted) deformation ring R is a complete intersection ring of expected dimension.

This conjecture is seemingly quite difficult, but fortunately it is entirely irrelevant for us; indeed, one of the advantages of the derived deformation ring \mathcal{R} is that this conjecture may sometimes be circumvented.

The derivedness of the rings we consider, then, most likely arises only at the next step, when we impose local conditions:

What is important in number theory is not just the bare Galois deformation ring, but the ring which classifies only lifts $\Gamma_S \rightarrow \mathrm{GL}_n(A)$ that are (suitably defined) “crystalline” (or similar; see §9). In our setting, this means that we represent not the functor $A \mapsto \mathrm{Def}_{\bar{\rho}}(A)$ but rather the homotopy fiber product

$$(1.6) \quad \mathrm{Def}_{\bar{\rho}} \times_{\mathrm{Def}_{\bar{\rho}_p}}^h \mathrm{Def}_{\bar{\rho}_p}^{\mathrm{crys}},$$

where $\mathrm{Def}_{\bar{\rho}_p}$ is the similarly defined deformation functor for the restriction $\bar{\rho}_p$ of $\bar{\rho}$ to the Galois group of \mathbb{Q}_p , and $\mathrm{Def}_{\bar{\rho}_p}^{\mathrm{crys}}$ represents the crystalline deformations for

$\bar{\rho}_p$. Recall (see Example A.4) that this homotopy fiber product fits into the diagram

$$(1.7) \quad \begin{array}{ccc} \mathrm{Def}_{\bar{\rho}} \times_{\mathrm{Def}_{\bar{\rho}_p}}^h \mathrm{Def}_{\bar{\rho}_p}^{\mathrm{crys}} & \longrightarrow & \mathrm{Def}_{\bar{\rho}_p}^{\mathrm{crys}} \\ \downarrow & & \downarrow \\ \mathrm{Def}_{\bar{\rho}} & \longrightarrow & \mathrm{Def}_{\bar{\rho}_p} \end{array}$$

and, explicitly it assigns to $A \in \mathrm{Art}_k$ a simplicial set whose (e.g) vertices correspond to a vertex of $\mathrm{Def}_{\bar{\rho}}(A)$, a vertex of $\mathrm{Def}_{\bar{\rho}_p}^{\mathrm{crys}}(A)$ and finally a path (i.e. a 1-simplex) between their images inside $\mathrm{Def}_{\bar{\rho}_p}(A)$.

For the moment, denote by $\mathcal{R}^{\mathrm{crys}}$ a ring that represents the functor (1.6). This is the derived deformation ring that is of primary interest to us.

In the most classically studied cases, for example, deformations of Galois representations for the modular curve, *the ring $\mathcal{R}^{\mathrm{crys}}$ will again be, under mild assumptions, homotopy discrete*: that follows by Lemma 7.5 and the fact that the usual crystalline deformation ring is known to be a complete intersection. In other words, the derived ring carries no extra information at all.

But for the general case (when one studies modular forms on SL_n for $n \geq 3$, for example – or for that matter if we consider *even* 2-dimensional Galois representations), $\mathcal{R}^{\mathrm{crys}}$ should not be expected to be homotopy discrete – it has higher homotopy groups. In the next section, we explain how it should relate to the homology of arithmetic groups.

We have already mentioned that $\pi_0 \mathcal{R} \cong R$. A second basic property of the deformation ring \mathcal{R} is that we can identify its André-Quillen cohomology with the cohomology of Γ_S , valued in the adjoint representation:

$$(\text{André-Quillen cohomology of } W(k) \rightarrow \mathcal{R} \text{ in degree } i) \cong H^{i+1}(\Gamma_S, \mathrm{Ad} \bar{\rho})$$

There are similar results for the ring $\mathcal{R}^{\mathrm{crys}}$ where we impose crystalline conditions.

To get a sense of the extra information in \mathcal{R} , let us discuss briefly $\pi_1 \mathcal{R}$, which is a module over $\pi_0 \mathcal{R}$. Roughly speaking, $\pi_1 \mathcal{R}$ is made of two pieces: the first is related to the failure of R to be complete intersection; and the second is related to the discrepancy between $\dim H^2(\Gamma_S, \mathrm{Ad} \bar{\rho})$ and the number of relations in a minimal presentation of R . The equation (4.7) gives a precise formulation of the prior sentence.

Comparison with naive construction using Tor-groups: It is well-known that “derived” rings arise naturally when one takes non-transversal intersections: informally, when one intersects the subvarieties $X, Y \subset Z$, the structure sheaf of $X \cap Y$ should be considered as a (homotopy) sheaf of simplicial commutative rings whose homotopy groups are given by $\mathrm{Tor}_*^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y)$. In the story above, derivedness has come from the non-transversal intersection between

$$X = \text{the moduli space of geometric representations of } \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$$

$$Y = \text{the moduli space of } \Gamma_S\text{-representations}$$

within

$$Z = \text{the moduli space of all representations of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).$$

Correspondingly, an easier-to-define version of the derived representation ring could be made by

$$(1.8) \quad \text{Tor}_*^{\mathbf{R}_p}(\mathbf{R}, \mathbf{R}_p^{\text{crys}}),$$

where $\mathbf{R}, \mathbf{R}_p, \mathbf{R}_p^{\text{crys}}$ are respectively the usual deformation ring, deformation ring at p , and the crystalline deformation ring at p . If we accept the Conjecture discussed after (1.5), this direct construction actually does give the homotopy groups $\pi_* \mathcal{R}^{\text{crys}}$. Unsurprisingly it is nonetheless helpful to work with the more refined version of the construction rather than this more concrete version; for example, we do not need to worry about the validity of the Conjecture (if the Conjecture is false, (1.8) will not have good formal properties).

1.4. Arithmetic results about the derived deformation ring. As mentioned in the abstract, the point of the number theory section is to describe how the ring $\mathcal{R}^{\text{crys}}$ arises naturally in the context of the Langlands program. More precisely, the Langlands program predicts (approximately speaking) that $\pi_0 \mathcal{R}^{\text{crys}}$ acts on the homology of arithmetic groups, and we show that the Taylor–Wiles method can be used to upgrade this action to a graded action of $\pi_* \mathcal{R}^{\text{crys}}$. To carry out this upgrading, we establish an isomorphism between $\pi_* \mathcal{R}^{\text{crys}}$ and a limit ring that arises in the Taylor–Wiles method (in the form [6] of Calegari–Geraghty).

Our results require some assumptions (Conjecture 6.1, similar to the conjectures assumed in [6]) on the existence of Galois representations and local-global compatibility, which are not known in general at present; we assume they are known in the discussion that follows.

We need to briefly summarize the setup of the Taylor–Wiles method. In the paragraphs that follows, we will be talking of *usual* (rather than derived) deformation rings, and will denote the usual rings by roman face, e.g. \mathbf{R} .

In the Taylor–Wiles method, we wish to study the deformation ring \mathbf{R} of a given residual representation $\overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(k)$, with G an algebraic group and k a finite field of characteristic p . This representation $\overline{\rho}$ arises, by means of the Langlands correspondence, from a Hecke eigenclass in the homology of some arithmetic manifold Y . The situation and our precise local and global assumptions are detailed more carefully in §13.1.

In our discussion we will always assume that $\overline{\rho}$ is crystalline at p . All our deformation rings will be those with crystalline conditions imposed at p , and we will therefore omit the superscript “crys” from our notation.

Let $W = W(k)$ be the ring of Witt vectors of k .

To proceed one considers the deformation ring \mathbf{R}_n after enlarging the set S , replacing it by $S \amalg Q_n$ for a suitable set of auxiliary primes.

Now there is a natural map $\mathbf{R}_n \rightarrow \mathbf{R}$. Moreover, by studying the action of inertia groups in S we produce a map $(\mathbb{Z}/p^n)^s \rightarrow \mathbf{R}_n^\times$, for suitable s ; thus \mathbf{R}_n is an algebra

over $S_n = W[(\mathbb{Z}/p^n)^s]$, the group algebra of $(W/p^n)^s$. Finally one can recover R by descent:

$$(1.9) \quad R = R_n \otimes_{S_n} W.$$

By a suitable limit process one can choose the rings R_n, S_n to approximate (i.e. be quotients of) limit rings R_∞, S_∞ that are of a very simple shape: *in the best situation (which we will suppose from now on)*, they are both formal power series rings over W , i.e.

$$R_\infty \simeq W[[x_1, \dots, x_n]], \quad S_\infty \simeq W[[y_1, \dots, y_{n+\delta}]]$$

for some integers n, δ ; and by taking the $n \rightarrow \infty$ limit of (1.9), we get

$$(1.10) \quad R = R_\infty \otimes_{S_\infty} W.$$

Because of the very simple shape of the limit rings, the formula (1.10) gives a lot of information about R .

Let us say a little bit more about how this argument works, since we will need some of its internal notation. When we say R_n approximates R_∞ , what we actually mean is that a certain Artinian quotient $R_n \twoheadrightarrow \overline{R}_n$ is isomorphic to a quotient of R_∞ , where both quotients become deeper and deeper as $n \rightarrow \infty$. There is a similar result to (1.9) using \overline{R}_n and $\overline{S}_n := S_n/p^n$ instead of R_n , and just recovering a certain quotient of R (this quotient will become closer and closer to all of R as n increases). One then obtains (1.10) by taking the $n \rightarrow \infty$ limit of this modified version of (1.9).

It was discovered by Calegari and Geraghty that not just the tensor product, but also the higher Tor groups, play an important role in the theory of modular forms. Their analysis implies the following:

(Calegari–Geraghty): The homology $H_*(Y)$ of the associated arithmetic manifold Y , localized at the ideal \mathfrak{m} of the Hecke algebra that corresponds to $\overline{\rho}$, is free over the graded ring $\mathrm{Tor}_*^{S_\infty}(R_\infty, W)$.

Our main result is that, in fact, these Tor-groups are captured by the derived deformation ring:

Main theorem with assumptions as above: Notations as above, with \mathcal{R} the derived deformation ring of $\overline{\rho}$ with crystalline conditions imposed at p , there is an isomorphism of graded rings

$$\pi_* \mathcal{R} \simeq \mathrm{Tor}_*^{S_\infty}(R_\infty, W).$$

From the result of Calegari–Geraghty above, it follows that

- (i) the localized homology $H_*(Y)_{\mathfrak{m}}$ carries the structure of a free graded module over $\pi_* \mathcal{R}$, and therefore
- (ii) $\pi_j \mathcal{R}$ is nonvanishing precisely for $0 \leq j \leq \delta$.

See Theorem 14.1 for the precise statement. Note that there are various choices made in the proof of Theorem 14.1, e.g., choices of subsequences to make compactness arguments, and *a priori* the module structure from (i) might depend on

all these choices. In fact it does not – this independence is proven only in the final section §15.

We now describe the proof, in outline.

The first step of the proof is to construct a map from \mathcal{R} to the derived tensor product $R_\infty \otimes_{S_\infty} W$. To do this, we use a compactness argument to extract a limit of the following maps:

$$(1.11) \quad \mathcal{R} \simeq \mathcal{R}_n \otimes_{S_n} W \rightarrow \pi_0 \mathcal{R}_n \otimes_{\pi_0 S_n} W \rightarrow \overline{R}_n \otimes_{\overline{S}_n} W / p^n$$

where \mathcal{R}_n, S_n are derived versions of R_n and S_n and \otimes is a derived version of tensor. The first isomorphism is a formality: it exhibits that “a representation of level $S \coprod Q_n$ unramified at Q_n is actually of level S .” The second is functoriality of the (derived) tensor product. The last map arises from $\pi_0 \mathcal{R}_n = R_n \rightarrow \overline{R}_n$ and similar for S .

Now, passing to the limit over n , we produce a map $\mathcal{R} \rightarrow R_\infty \otimes_{S_\infty} W$. We check it is an isomorphism by checking the induced map on André–Quillen cohomology. Both sides have tangent complexes supported only in two dimensions, and it’s clear it’s an isomorphism in one degree; to verify in the other degree, we check the induced map is surjective and then compare Euler characteristics.

Although the implications (i) and (ii) mentioned in the statement of the Theorem don’t seem to be accessible without proving the full result, other implications of the Theorem are obvious or can be obtained more directly. Let us talk through these to give some orientation of where the content is.

- (a) The theorem implies that \mathcal{R} (at least at the level of homotopy groups) is obtained by taking n free generators and imposing (in the derived sense) $n + \delta$ relations. This can be deduced directly from general facts of deformation theory (cf. Corollary 4.5), where n is the dimension of a suitable tangent space.
- (b) The theorem implies that, when $\delta = 0$, \mathcal{R} is homotopy discrete, i.e. the natural map $\mathcal{R} \rightarrow \pi_0 \mathcal{R}$ is an isomorphism. This can be deduced directly (Lemma 7.5) if one knows that $\pi_0 \mathcal{R}$ is a complete intersection of the expected size. When $\delta = 0$ one can usually get this from the usual Taylor–Wiles method [39, 34]; one does not really need to go through the main theorem.
- (c) The theorem also allows to compute the homotopy groups in characteristic zero: a geometric lift $\tilde{\rho}$ of the original Galois representation to W gives a homomorphism $\pi_0 \mathcal{R} \rightarrow W$; writing E for the quotient field of $W(k)$, the associated ring $\pi_* \mathcal{R} \otimes_{\pi_0 \mathcal{R}} E$ is isomorphic to the exterior algebra of a δ -dimensional vector space over E . Again this can be deduced directly without much trouble (at least, assuming vanishing of $H_f^1(\text{Ad} \tilde{\rho})$, which is a consequence of standard conjectures).

On the other hand, neither of the noted implications (i) or (ii) are obvious on general grounds. In particular the statement about the free action on homology seems to be the key point.

To conclude, we study in §15 the relationship between \mathcal{R} and the derived Hecke algebra introduced in [37]. The derived Hecke algebra and the derived representation ring seem to be of different natures; the Hecke algebra acts on cohomology, increasing cohomological degree, and \mathcal{R} on homology, increasing homological degree. The relationship is as follows: one looks like the exterior algebra on a vector space V and the other looks like the exterior algebra on V^* . The eventual result (Theorem 15.2) shows, in particular, that *the action of $\pi_*\mathcal{R}$ is in fact independent of the choice of sequence of Taylor–Wiles primes used.*

We also mention the related paper [16] of Hansen and Thorne. There an action of an exterior algebra on homology is produced by (roughly speaking) adding level at p (rather than Taylor–Wiles level) and descending. It would be useful to identify this exterior algebra also with $\pi_*\mathcal{R}$.

Remark 1.1. *Our various assumptions on $\overline{\rho}$ from §13.1 – in particular, excluding congruences with other forms – have the effect of also forcing $\pi_*\mathcal{R}$ to be an integral exterior algebra. As stated, the results of §15 would not be true if this were not the case. If we dropped these simplifying assumptions, we don’t expect $\pi_*\mathcal{R}$ to be an integral exterior algebra, but nonetheless we would expect that the comparison results of §15 should remain true after tensoring with \mathbb{Q} .*

By contrast, the statement that the localized homology $H_(Y)_{\mathfrak{m}}$ carries the structure of a free $\pi_*\mathcal{R}$ -module doesn’t use the full strength of the assumptions on $\overline{\rho}$ – in particular, it only uses “minimal level.” We expect that it continues to be valid, even integrally, even without that assumption.*

1.5. Overview of paper and suggestions for reading. The first part of the paper is generalities that are not specific to number theory: It uses freely the language of model categories. An introduction to this language may be found in [12]. We also use freely the language of homotopy limits and colimits, because we need to work with *pro*-representable functors. A brief review of this language is given in Appendix A.

§2 is an overview of basic results about Artin local simplicial commutative rings and simplicial functors on this category; we define representability and prove a general result concerning the approximation of a functor by a representable functor.

§4 reviews the tangent complex of a functor and Lurie’s derived Schlessinger criterion, and develops some tools for manipulating and understanding (pro-)representable functors and their representing objects.

§5 defines the functors that we are interested in representing, namely, the derived space of homomorphisms from a profinite group Γ to $G(A)$, where G is an algebraic group and A a simplicial ring. The definitions here require some care.

The remainder of the paper studies the specific case of deformations of a Galois representation:

§6 and §7 collect the notation to be used in the number theory sections (§6 collects notation about Galois representations and cohomology, and §7 summarizes notations about derived deformation rings).

§8 examines what happens to the deformation ring when we add a single prime to the ramification set. (The result is intuitively obvious.) The case of main interest is when this prime is a “Taylor–Wiles prime,” thus the title of the section.

§9 discusses how to impose “local conditions” on the derived deformation ring. The point of main interest to us is imposing a crystalline condition.

§11 gets down to business: The Taylor–Wiles method involves adding a set Q of ramified primes with carefully chosen cohomological properties. In §11 we show that these cohomological properties give a tight control on the derived deformation ring after allowing ramification at Q .

§12 gives an abstract discussion of how to extract limits of maps like (1.11).

§13 summarizes the obstructed Taylor–Wiles method, as developed by Calegari–Geraghty; we use the formulation of Khare and Thorne.

§14 proves the main theorem, and §15 gives the comparison between the action of the derived deformation ring and the derived Hecke algebra.

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2. FUNCTORS OF SIMPLICIAL ARTIN RINGS

We shall assume the reader is familiar with standard properties of the category of simplicial sets, which we shall denote $s\mathbf{Sets}$, including the notion of *Kan complexes* and the fibrant replacement $X \rightarrow \mathrm{Ex}^\infty(X)$, as well as the self-enrichment of $s\mathbf{Sets}$: the set of morphisms $X \rightarrow Y$ between two simplicial sets X and Y forms the 0-simplices in a simplicial set $s\mathbf{Sets}(X, Y)$. This has the expected behavior when Y is Kan (e.g. homotopy equivalences $X' \rightarrow X$ and $Y \rightarrow Y'$ induce a homotopy equivalence $s\mathbf{Sets}(X, Y) \rightarrow s\mathbf{Sets}(X', Y')$ when Y and Y' are Kan). When the distinction between the set of maps and the simplicial set of maps is important, we shall write $s\mathbf{Sets}_0(X, Y)$ for the set of maps.

As usual, we denote by $|X|$ the geometric realization of the simplicial set X , and by $\mathrm{Sing}(Y)$ the simplicial set of simplices associated to a topological space Y . We shall write Δ^q or $\Delta[q]$ for the usual simplex $[p] \mapsto \Delta([p], [q])$ and $\partial\Delta^q = \partial\Delta[q]$ for its boundary. If $X = (X, x_0)$ is a pointed simplicial set, we denote by ΩX the simplicial loop space, i.e. the simplicial set $s\mathbf{Sets}((\Delta^1, \partial\Delta^1), (X, x_0))$ of pointed maps from the simplicial circle to X .

2.1. Simplicial commutative rings. Let us write \mathbf{SCR} for the category of simplicial commutative rings, i.e. functors from Δ^{op} to commutative rings. Recall from [28, II, §4] that \mathbf{SCR} comes with subcategories of weak equivalences, cofibrations and fibrations satisfying the axioms of a *simplicial model category*. We refer there for more details, but briefly recall some of the key definitions.

For an object $R \in \text{SCR}$ we write $R^{\Delta[p]} = s\text{Sets}(\Delta^p, R)$, which is naturally an object of SCR . If $R' \in \text{SCR}$ is another object, the set of morphisms $R' \rightarrow R^{\Delta[p]}$ is the p -simplices of a simplicial set $\text{SCR}(R', R)$ of maps, and in this way SCR is enriched over $s\text{Sets}$.

For an object $R \in \text{SCR}$, the homotopy groups $\pi_*(R) = \bigoplus_n \pi_n(|R|, 0)$ form a graded commutative ring and a morphism $R \rightarrow R'$ in SCR is a weak equivalence if it is a weak equivalence of underlying simplicial sets, i.e. induces an isomorphism on all homotopy groups. A morphism $R' \rightarrow R$ in SCR is a *fibration* if the underlying morphism of simplicial sets is a Kan fibration, and we recall that this is automatic when the map of p -simplices $R'_p \rightarrow R_p$ is surjective for all p (in fact happens if and only if the restriction to the components containing 0 is surjective in each simplicial degree).

Finally, *cofibrations* $R' \rightarrow R$ are defined by a lifting property, but we shall recall a particular source of cofibrations which play an important role in this paper. If X is a set we shall write $\mathbb{Z}[X]$ for the free commutative algebra on X and if X is a simplicial set we shall use the same notation $\mathbb{Z}[X]$ for the simplicial commutative ring arising by applying this construction in each simplicial degree. Then if $R \in \text{SCR}$ and $e : \partial\Delta^p \rightarrow R$ is a morphism of simplicial sets, there is a unique extension $\mathbb{Z}[\partial\Delta^p] \rightarrow R$ to a morphism in SCR , and we obtain a morphism

$$(2.1) \quad R \rightarrow R' = R \otimes_{\mathbb{Z}[\partial\Delta^p]} \mathbb{Z}[\Delta^p]$$

in SCR , whose target depends on the *attaching map* $e : \partial\Delta^p \rightarrow R$ even though we omitted it in the notation. (In the above equation, the tensor product is taken level-wise.)

Definition 2.1. *The simplicial commutative ring obtained from R by attaching a cell along $e : \partial\Delta^p \rightarrow R$ is the simplicial commutative ring R' defined by (2.1).*

For $p = 0$ this just amounts to adjoining a single polynomial generator in each simplicial degree. For any p and e the resulting map $R \rightarrow R'$ is a cofibration, as are finite or transfinite compositions of maps of this form.

Then an arbitrary morphism $R'' \rightarrow R$ may be factored as $R'' \rightarrow R' \rightarrow R$ where $R'' \rightarrow R'$ is a cofibration and $R' \rightarrow R$ is both a weak equivalence and a fibration. The existence of such a factorization is part of the axioms of a model category, but a particular proof of its existence constructs R' as a (possibly transfinite) composition of cell attachments. In the special case where $R'' = \mathbb{Z}$ is the initial object we obtain a *cofibrant approximation* $R' \rightarrow R$ where R' is obtained from \mathbb{Z} by iterated cell attachments, similar to “CW approximations” of topological spaces. If we don’t attempt to control the cardinality of the set of cell attachments it is possible to construct $R' \rightarrow R$ as a functor of R , a *functorial cofibrant approximation*. We shall pick one such and denote it $c(R) \rightarrow R$ or sometimes $R^c \rightarrow R$ for typographical convenience.

Later in the paper we shall discuss the analogue of “minimal CW approximations” of topological spaces: roughly speaking we may for a given R ask for a weak equivalence $R' \rightarrow R$ where R' is built using the minimal possible number of

cells of each dimension. Such a minimal $R' \rightarrow R$ will not be functorial in R , but turns out to exist, at least when SCR is replaced by a modified category “pro- Art_k ” which we shall also define later.

2.2. Simplicial Artin rings. Recall that k is a fixed (usually finite) field, which we regard as an object of SCR and write $\text{SCR}/_k$ for the over category. Following the setup of [22], we shall be especially interested in a certain full subcategory $\text{Art}_k \subset \text{SCR}/_k$ of “Artin” objects, defined in the next subsection. In itself it is too small to be a model category, since it does not have enough limits and colimits (e.g. it has no initial object) – but we shall study its homotopy theory using the forgetful functors $\text{Art}_k \rightarrow \text{SCR}/_k \rightarrow \text{SCR}$. By a mild abuse of language we shall say e.g. “ $R \in \text{Art}_k$ is cofibrant” to mean that “ $R \in \text{Art}_k$ is has cofibrant image under the inclusion functor $\text{Art}_k \rightarrow \text{SCR}/_k$ ”, etc.

Definition 2.2. *An object $A \in \text{SCR}$ is Artin local if $\pi_0 A$ is Artin local in the usual sense and $\pi_* A = \bigoplus_n \pi_n A$ is finitely generated as a module over $\pi_0 A$. For a (usually finite) field k , the category $\text{Art}_k \subset \text{SCR}/_k$ is the full subcategory whose objects are the $\epsilon : A \rightarrow k$ with A Artin local and $\epsilon : \pi_0(A) \rightarrow k$ is surjective. (In other words, ϵ is a specified isomorphism from the residue field of $\pi_0(A)$ to k .)*

For typographical reasons we shall often denote the object $(\epsilon : A \rightarrow k) \in \text{Art}_k$ by simply A , but we emphasize that the map to k is part of the data and that morphisms in Art_k are required to commute with the maps to k .

Lemma 2.3. *If $B \rightarrow D \leftarrow C$ is a diagram in Art_k such that either $B \rightarrow D$ or $C \rightarrow D$ is surjective in each simplicial degree, then the fiber product $A = B \times_D C$ is also an object of Art_k .*

Proof. If one of the maps is degreewise surjective then it is a fibration and the map to the homotopy fiber product $B \times_D C \rightarrow B \times_D^h C$ is a weak equivalence. (See Example A.4 for definition of the homotopy fiber product.) Therefore, the homotopy groups of A fit into a Mayer–Vietoris short exact sequence with those of B , C , and D . This is a sequence of modules over the ring $\pi_0(A)$, from which it is easily deduced that $\pi_0(A)$ is Artin local in the usual sense. The finite-length condition also follows from the Mayer–Vietoris sequence. \square

Definition 2.4. (i) *If V is a simplicial k -module, the object $k \oplus V \in \text{SCR}/_k$ is defined by square-zero extension in each simplicial degree. This is an object of Art_k if and only if $\dim_k(\pi_*(V)) < \infty$.*
(ii) *For $n \geq 0$ write $S^n = \Delta^n / \partial \Delta^n$ for pointed simplicial set obtained by collapsing the boundary of the simplex to a point. Then write $k[n]$ for the free simplicial k -module generated by S^n (i.e. with p -simplices the free k -module on the p -simplices of S^n modulo the span of the basepoint). Write $k \oplus k[n]$ for the corresponding square-zero extension.*
(iii) *More generally for a k -module V , write $V[n] = V \otimes_k k[n]$ for the simplicial k -module obtained as the tensor product in each simplicial degree, and $k \oplus V[n]$ for the square-zero extension.*

(iv) Let $\widetilde{k[n]} = \{0\} \times_{k[n]}^h k[n]$ be the homotopy fiber product of the diagram $0 \rightarrow k[n] \leftarrow k[n]$, which is again a simplicial k -module, and $k \oplus \widetilde{k[n]}$ for the corresponding square-zero extension. Similarly for $V[n]$ and $k \oplus V[n]$.

The notation $V[n]$ is inspired by the corresponding notation from chain complexes: the homotopy groups of $V[n]$ are concentrated in degree n and $\pi_n(V[n], 0)$ is canonically isomorphic to V .

The homotopy fiber product in (iv) is contractible and has the property that $\widetilde{k[n]} \rightarrow k[n]$ is a Kan fibration. (In fact any simplicial k -module with these properties would work just as well as $\widetilde{k[n]}$ for what follows.) Similarly, $k \oplus \widetilde{k[n]} \rightarrow k \oplus k[n]$ can be viewed as just a particular way of replacing the unique morphism $k \rightarrow k \oplus k[n]$ in Art_k by a fibration.

The objects $k \oplus k[n] \in \text{Art}_k$ play a special role, due to the following special case of the pullback construction.

Example 2.5. Let $h : A \rightarrow k \oplus k[n]$ be any morphism in Art_k , and define $A' \rightarrow A$ by the pullback diagram (pullback in each simplicial degree)

$$(2.2) \quad \begin{array}{ccc} A' & \longrightarrow & k \oplus \widetilde{k[n]} \\ \downarrow & & \downarrow \\ A & \xrightarrow{h} & k \oplus k[n], \end{array}$$

Then, for any $f : B \rightarrow A$, the space of null homotopies of the composition $B \rightarrow A \rightarrow k \oplus k[n]$ is isomorphic to the space of lifts of f to $B \rightarrow A'$.

The word “null homotopies” in the above example should be interpreted as “paths to the trivial homomorphism through ring homomorphisms”, in the following way. The composition $h \circ f$ is a 0-simplex in $\text{Art}_k(B, k \oplus k[n])$ as is the composition $B \rightarrow k \rightarrow k \oplus k[n]$ of the augmentation map and the unique morphism $k \rightarrow k \oplus k[n]$. These two compositions assemble to one map $\partial\Delta^1 \rightarrow \text{Art}_k(B, k \oplus k[n])$ and by the space of null homotopies we mean the simplicial set of extensions to maps $\Delta^1 \rightarrow \text{Art}_k(B, k \oplus k[n])$. By “space of lifts” we mean the simplicial subset $s\text{Sets}(B, A')$ consisting of maps making the triangle commute. These are isomorphic by the definition of the homotopy fiber product used in defining $\widetilde{k[n]}$.

One reason for the importance of the objects $k \oplus k[n]$ is that any object or morphism in Art_k is weakly equivalent to one built by finitely many iterations of the process explained in Example 2.5 above. Let us establish a few preliminary properties of the objects $k \oplus V[n]$.

Lemma 2.6. Let $n \geq 0$ and let V and W be finite-dimensional k -vector spaces. Let R be an object of Art_k and $\phi : \pi_*(R) \rightarrow \pi_*(k \oplus V[n])$ an isomorphism of graded rings. Then ϕ is induced by a zig-zag $R \leftarrow R' \rightarrow k \oplus V[n]$ of weak equivalences in Art_k .

Proof. First use that $\mathbb{Z} \rightarrow k$ is complete intersection, so there exists a cofibrant model R'' of k as a \mathbb{Z} -algebra built using generators in degree 0 and 1 only: indeed, the ring k can be obtained from $\mathbb{Z}[x]$ by killing the regular sequence $(p, f(x))$, where f is any integral lift of an irreducible polynomial in $\mathbb{F}_p[x]$ of appropriate degree, and we may build R'' by attaching two 1-cells to $\mathbb{Z}[x]$ along p and $f(x)$. Then there are no obstructions to finding a map $R'' \rightarrow R$ inducing isomorphisms in π_k for $0 \leq k < n$. Then pick a generating set $v_1, \dots, v_d \in V$ and write $R''[v_1, \dots, v_d]$ for the simplicial commutative ring obtained from R'' by attaching one n -cell along the constant map $0 : \partial\Delta^n \rightarrow R''$ for each basis element, in the sense of Definition 2.1. The morphism $R'' \rightarrow R$ then extends to a morphism $R''[v_1, \dots, v_d] \rightarrow R$ inducing a bijection in π_k for $0 \leq k < n$ and a surjection for $k = n$. If we then define $R''[v_1, \dots, v_d] \rightarrow R'$ by attaching cells of dimension $n + 1$ and higher to kill generators for the kernel in π_n and all higher homotopy groups, there is no obstruction to extend to a morphism $R' \rightarrow R$ which will then be a weak equivalence. The same argument also applies to give a weak equivalence $R' \rightarrow k \oplus V[n]$, giving the desired zig-zag. \square

Remark 2.7. *It is not quite true that morphisms $k \oplus V[n] \rightarrow k \oplus W[n]$ in Art_k are “the same thing” as k -linear maps $V \rightarrow W$, even up to homotopy. For example for $n = 1$, $V = 0$ and $W = k$, the space $\text{Art}_k(k, k \oplus k[1])$ is not path connected, and in fact has $\pi_0 = \pi_{-1}\text{tk} \cong k$. Morally, the reason for this is that $\text{Art}_k(A^c, B) = \text{SCR}_{/k}(A^c, B)$ is the space of derived morphisms of \mathbb{Z} -algebras; when A and B happen to be represented by simplicial k -algebras there is also a notion of a derived space of k -algebra maps, but it will have a different homotopy type.*

Let us also briefly discuss *tensor products* of simplicial commutative rings. Given a diagram $R' \leftarrow R \rightarrow R''$ we may form the tensor product $R' \otimes_R R''$ levelwise, and this inherits the usual universal property from commutative rings: a homomorphism out of it is the same as a pair of homomorphisms out of R' and R'' restricting to the same homomorphism out of R . In order to obtain a homotopy invariant version of this construction, it should only be applied when either R' or R'' is cofibrant as an R -algebra (e.g. it is built from R using a finite or transfinite iteration of cell attachments). In that situation there is a spectral sequence ([28, Theorem 6, §6, II])

$$E^2 = \text{Tor}_{\pi_*(R)}(\pi_*(R'), \pi_*(R'')) \Rightarrow \pi_*(R' \otimes_R R'').$$

In particular suppose $R \rightarrow R'$ is a cofibration in Art_k and $\pi_m(R', R) = 0$ for all $m \leq n - 1$. In that case the spectral sequence implies the isomorphism $\pi_n(R' \otimes_R k) = \pi_n(R', R) \otimes_{\pi_0 R} k$.

Lemma 2.8. *(i) Let $R \rightarrow k$ be an object of $\text{SCR}_{/k}$. Then $R \rightarrow k$ is an object of Art_k if and only if there exists a sequence of cofibrant simplicial commutative rings A_0, A_1, \dots, A_m with $A_0 = k$ and a weak equivalence $R \rightarrow A_m$, where A_i is a cofibrant replacement of a simplicial commutative ring A'_{i-1} obtained from A_{i-1} by a pullback along $h_i : A_{i-1} \rightarrow k \oplus k[n_i]$ as above, with all $n_i \geq 1$.*

- (ii) Any morphism $R \rightarrow A$ in Art_k (with R cofibrant) may be factored as $R \rightarrow A' \rightarrow A$, where the first map $R \rightarrow A'$ is a weak equivalence and the second map $A' \rightarrow A$ is obtained by composing a finite sequence of pullback diagrams as in (2.2) (and cofibrant replacement) with varying $n \geq 0$.

Proof sketch. For (i) see also [22, Lemma 6.2.6].

We prove (ii). We will use Postnikov truncations, for which we refer the reader ahead to §3.1. Without loss of generality we may assume that $R \rightarrow A$ is a cofibration. Suppose in addition that it is $(n-1)$ -connected, i.e. that the relative homotopy groups $\pi_*(A, R)$ vanish in degrees strictly below $n-1$, for some $n \geq 1$, and consider the map from A to the tensor product $A \otimes_R k$. Then $\pi_n(A \otimes_R k) = \pi_n(A, R) \otimes_{\pi_0 R} k = V$ for some k -vector space V . Then the truncation $\tau_{\leq n}(A \otimes_R k)$ has the same homotopy groups as $k \oplus V[n]$ and hence for R and A cofibrant there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & k \oplus \widetilde{V[n]} \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus V[n] \end{array}$$

such that the induced map $\pi_m(A, R) \rightarrow \pi_m(k \oplus V[n], k)$ is an isomorphism for $m < n$ and surjective for $m = n$. If we write $R \rightarrow A'$ for the induced homomorphism from R to the pullback of the rest of the diagram, then we conclude that this map remains $(n-1)$ -connected and that the group $\pi_n(A', R)$ is identified with the kernel of the quotient map $\pi_n(A, R) \rightarrow \pi_n(A, R) \otimes_{\pi_0 R} k \cong \pi_n(A, A') = V$. Since $\pi_n(A, R)$ has finite length as a module over $\pi_0 R$ we may repeat this process finitely many times to achieve $\pi_n(A', R) = 0$ so that $R \rightarrow A'$ is n -connected. Then continue by induction. This finishes (ii) in the case where $\pi_0(R) \rightarrow \pi_0(A)$ is surjective, and in particular proves (i).

If $\pi_0(R) \rightarrow \pi_0(A)$ is not surjective there is an easy preliminary step first, using $n = 0$. \square

The first part of the above lemma has the following analogy in pointed spaces. A pointed connected space is p -finite if it has only finitely many non-zero homotopy groups, all of which are finite p -groups. Then a pointed connected space is p -finite if and only if it is weakly equivalent to one obtained from a point by finite iterations replacing a pointed space X by the homotopy fiber of a map $h : X \rightarrow K(\mathbb{F}_p, n)$, with varying $n \geq 2$. The second part of the above lemma has a similar analogy for maps between p -finite spaces. (We find the analogy to p -finite spaces instructive and shall return to it later in the paper.)

2.3. Functors and natural transformations. We shall study functors $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ and natural transformations between them.

Definition 2.9. A natural simplicial homotopy between two natural transformations $S, T : \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation $\Delta[1] \times \mathcal{F}(-) \rightarrow \mathcal{G}$ giving a homotopy between $S, T : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ for all A .

From this definition of simplicial homotopy one could define a notion of “natural simplicial homotopy equivalence” in terms of functors in both directions and homotopies from the two compositions to the two identity natural transformations, but that would be a very strict notion rarely satisfied in practice. The following weaker notion is much more useful.

Definition 2.10. *Let $\mathcal{F}, \mathcal{G} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ be two functors. A natural weak equivalence is a natural transformation $T : \mathcal{F} \rightarrow \mathcal{G}$ inducing a weak equivalence $\mathcal{F}(A) \rightarrow \mathcal{G}(A)$ for all $A \in \mathbf{Art}_k$. Two functors are naturally weakly equivalent if there exists a (finite) zig-zag of natural weak equivalences between them.*

For example, any functor is naturally weakly equivalent to one taking values in Kan simplicial sets. Indeed, either of the natural transformations $\mathcal{F}(A) \rightarrow \mathrm{Sin}|\mathcal{F}(A)|$ or $\mathcal{F}(A) \rightarrow \mathrm{Ex}^\infty(A)$ is a natural weak equivalence, where Ex^∞ is Kan’s infinite iteration of the adjoint subdivision. We shall be mostly concerned with functors having the following extra property.

Definition 2.11. *A functor $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ is homotopy invariant if $\mathcal{F}(\phi) : \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ is a weak equivalence for any weak equivalence $\phi : A \rightarrow B$ in \mathbf{Art}_k .*

Definition 2.12. *A simplicial enrichment of a functor $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ is the specification of maps of simplicial sets $\mathbf{Art}_k(A, B) \rightarrow \mathbf{sSets}(\mathcal{F}A, \mathcal{F}B)$ for all objects A, B , agreeing with the functoriality on 0-simplices and compatible with the compositions in the \mathbf{sSets} -enriched categories \mathbf{Art}_k and \mathbf{sSets} .*

A simplicially enriched functor is a functor together with a simplicial enrichment.

Lemma 2.13. *Any homotopy invariant functor $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ is naturally weakly equivalent to a simplicially enriched one. In fact, there is a simplicially enriched functor \mathcal{F}' with Kan values and a natural weak equivalence $\mathcal{F} \rightarrow \mathcal{F}'$.*

Proof sketch. We briefly give the construction: for an object $A \in \mathbf{Art}_k$ we write $A^{\Delta[p]} = \mathrm{Hom}_{\mathbf{sSets}}(\Delta[p], A)$. For fixed $[p]$, this simplicial set is canonically a simplicial ring. Since A is automatically fibrant as a simplicial set, the canonical map $A \rightarrow A^{\Delta[p]}$ is a weak equivalence of simplicial sets and in particular $A^{\Delta[p]}$ is again an object of \mathbf{Art}_k . Let $F'(A)$ be the diagonal simplicial set of the bisimplicial set $[p] \mapsto F(A^{\Delta[p]})$, i.e. the p -simplices of $F'(A)$ are the p -simplices of $F(A^{\Delta[p]})$. The canonical map of simplicial sets $F(A) \rightarrow F'(A)$ is a weak equivalence and it is possible to simplicially enrich F' in a natural way. For further details see [29, Corollary 6.5]. \square

Thus, when proving a statement of the form that a certain homotopy invariant functor \mathcal{F} is naturally weakly equivalent to another functor with certain properties, we may without loss of generality assume that \mathcal{F} takes values in Kan complexes and is simplicially enriched.

2.4. Representable functors. Following Schlessinger and Lurie, we shall be interested in functors which are *representable* and functors which are *pro-representable*.

Definition 2.14. A functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ is *representable* if it is naturally weakly equivalent to $\mathrm{Hom}(R, -)$ for some cofibrant object $R \in \mathbf{Art}_k$.

When \mathcal{F} is simplicially enriched, there is a “simplicial Yoneda-lemma”: sending a natural transformation $T : \mathrm{Hom}(R, -) \rightarrow \mathcal{F}$ of functors $\mathbf{Art}_k \rightarrow s\mathbf{Sets}$ to the zero-simplex $T(\mathrm{id}) \in \mathcal{F}(R)$ gives a bijection between such natural transformations and zero-simplices of $\mathcal{F}(R)$. In the same way, natural simplicial homotopies $\Delta[1] \times \mathrm{Hom}(R, -) \rightarrow \mathcal{F}$ correspond to 1-simplices of $\mathcal{F}(R)$, etc.

Lemma 2.15. A simplicially enriched functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ is *representable* if and only if there exists a cofibrant $R \in \mathbf{Art}_k$ and $\iota \in \mathcal{F}(R)$ such that the induced $\mathrm{Hom}(R, -) \rightarrow \mathcal{F}$ is a natural weak equivalence.

A general (possibly unenriched) representable functor \mathcal{F} is automatically homotopy invariant because $\mathrm{Hom}(R, -)$ is homotopy invariant for any cofibrant $R \in \mathbf{SCR}$. (This follows, e.g., from [17, Lemma 1.1.12], together with the fact that all simplicial rings are fibrant.) Therefore, by Lemma 2.13, we can always find a zig-zag $\mathcal{F} \rightarrow \mathcal{F}' \leftarrow \mathrm{Hom}(R, -)$ of length two.

Proof. It suffices to prove that if $S : \mathcal{F} \rightarrow \mathrm{Hom}(R, -)$ is a natural weak equivalence, then there exists a natural transformation $T : \mathrm{Hom}(R, -) \rightarrow \mathcal{F}$ such that ST is naturally simplicially homotopic to the identity natural transformation.

To construct such a T , pick $\iota \in \mathcal{F}(R)$ such that $S(\iota) \in \mathrm{Hom}(R, R)$ is in the same path component as the identity map and let T be the corresponding natural transformation. Then ST is the natural transformation corresponding to $S(\iota) \in \mathrm{Hom}(R, R)$. Since this is a Kan complex, there is a 1-simplex $\lambda : \Delta[1] \rightarrow \mathrm{Hom}(R, R)$ connecting the identity with $S(\iota)$; the adjoint $\lambda \in \mathrm{Hom}(R, R^{\Delta[1]})$ classifies a natural transformation $\Delta[1] \times \mathrm{Hom}(R, -) \rightarrow \mathrm{Hom}(R, -)$ from the identity to ST . \square

2.5. Approximation by representable functors. Representable functors play a central role in this paper, and we shall need criteria for which functors are representable. Let us first discuss some more simple-minded methods for “approximating” an *arbitrary* homotopy invariant, simplicially enriched, Kan valued functor $F : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ by representable ones. Our method shall be a homotopy version of the following strategy. Suppose C is an essentially small category which has all small limits and $F : C \rightarrow \mathbf{Sets}$ is a functor. Then there is a category whose object are pairs (c, ϕ) where c is an object of C and $\phi : C(c, -) \rightarrow F$ is a natural transformation; the morphisms $(c, \phi) \rightarrow (c', \phi')$ are pairs of a morphism $c' \rightarrow c$ and a natural isomorphism between the two resulting functors $\mathrm{Hom}(c', -) \rightarrow F$. Then we may to each such functor F canonically associate an object $c_F = \lim_{(c, \phi)} c$, and natural transformations

$$C(c_0, -) \leftarrow \mathrm{colim}_{(c, f)} C(c, -) \rightarrow F,$$

which are natural isomorphisms if F happens to be representable. In particular this formula extracted a representing object c_0 directly in terms of the functor F .

We shall follow a similar procedure to extract a representing object $R \in \text{Art}_k$ from a representable functor $F : \text{Art}_k \rightarrow s\text{Sets}$. First, we must replace all limits and colimits by homotopy limits and homotopy colimits. Second, the category Art_k is not essentially small in the strict sense, but it is “homotopy small” (there is a set of objects representing all homotopy classes). Third, the category of pairs (c, ϕ) above should be replaced by a *simplicial category* and the homotopy (co)limits should take this into account. In appendix A we discuss only homotopy (co)limits indexed by ordinary categories, so we shall not use this terminology here; instead we write explicit formulas. (For example the simplicial functor defined in Definition 2.16 below is the simplicial version of the formula $\text{colim}_{(S, \phi)} C(S, A) \rightarrow F(A)$ for functors into sets.)

Definition 2.16. *Pick, “once and for all” a set of cofibrant objects of Art_k , such that any object $R \in \text{Art}_k$ admits a weak equivalence from an object in the set, and write $\text{Art}_k^{\text{skel}} \subset \text{Art}_k$ for the full subcategory with these objects.*

Let $F : \text{Art}_k \rightarrow s\text{Sets}$ be homotopy invariant, simplicially enriched, and Kan valued. For $[p] \in \Delta$, let $M_p^F : \text{Art}_k \rightarrow s\text{Sets}$ be given by

$$M_p^F(A) = \coprod_{\substack{(S_0, \dots, S_p) \\ \in (\text{Art}_k^{\text{skel}})^{p+1}}} F(S_0) \times \text{Art}_k(S_0, S_1) \times \cdots \times \text{Art}_k(S_{p-1}, S_p) \times \text{Art}_k(S_p, A)$$

Composition of morphisms and functoriality of F define face maps $M_p^F(A) \rightarrow M_{p-1}^F(A)$ and insertion of identities define degeneracy maps. The diagonal of the resulting bisimplicial set shall be denoted also by $M^F : \text{Art}_k \rightarrow s\text{Sets}$.

More conceptually, there is a category $(\text{Art}_k^{\text{skel}} \wr F)$ whose objects are pairs (S, x) , $S \in \text{Art}_k^{\text{skel}}$ and $x \in F(S)$, and whose morphisms are $\phi : S_0 \rightarrow S_1$ send $x_0 \rightarrow x_1$. The category is the zero simplices of a *simplicial category* (i.e. simplicial object in the category of small categories) whose p -simplices is the category with objects pairs (S, x) with $x : \Delta[p] \rightarrow F(S)$ and morphisms $(S, x_0) \rightarrow (S, x_1)$ are the $\phi : \Delta[p] \rightarrow \text{Art}_k(S_0, S_1)$ with the property that the composition

$$\Delta[p] \xrightarrow{(x, \phi)} F(S_0) \times \text{Art}_k(S_0, S_1) \rightarrow F(S_1)$$

is x_1 . There is a simplicial functor $(\text{Art}_k^{\text{skel}} \wr F)^{\text{op}} \rightarrow s\text{Sets}$ given on objects by $(S, x) \mapsto \text{Art}_k(S, A)$, and the above construction of $M^F(A)$ can be regarded as the homotopy colimit of that functor, following a natural generalization of the Bousfield–Kan formula to the case where the indexing category is a simplicial category.

Lemma 2.17. *The evaluation map $M_0^F(A) = \coprod_{S \in A} F(S) \times \text{Art}_k(S, A) \rightarrow F(A)$ gives rise to a map*

$$M^F(A) \rightarrow F(A)$$

which is a natural weak equivalence.

Proof sketch. Since F and M^F are both homotopy invariant, it suffices to consider $A \in \text{Art}_k^{\text{skel}}$. If we write $M_{-1}^F(A) = F(A)$ the resulting augmented simplicial space has an “extra degeneracy” $s_{-1} : M_{p-1}^F(A) \rightarrow M_p^F(A)$ for each fixed A , defined by letting $S_p = A$ and inserting the identity on A . Hence the augmentation map is a weak equivalence (cf. e.g. [13, Lemma III.5.1]). \square

The universal strictly representable functor which admits a map from M^F is then the functor represented by the “homotopy limit” of the functor $(\text{Art}_k^{\text{skel}} \wr F) \rightarrow s\text{Sets}$ given on object as $(S, x) \mapsto S$, as we shall now explain. (This is entirely analogous to the discussion of homotopy limits in Appendix A, except now the indexing category $\text{Art}_k^{\text{skel}} \wr F$ is a simplicial category, and so one needs to take this into account.) There is a functor $(\text{Art}_k^{\text{skel}} \wr F) \rightarrow s\text{Sets}$ described on objects as

$$(S, x) \mapsto N((S, x) \downarrow (\text{Art}_k^{\text{skel}} \wr F)^{\text{op}}).$$

This is simplicial functor and we shall write it as $N(- \downarrow (\text{Art}_k^{\text{skel}} \wr F)^{\text{op}})$. We shall write

$$R^F \subset \prod_S s\text{Sets}(N(S \downarrow (\text{Art}_k^{\text{skel}} \wr F)^{\text{op}}), S)$$

for the simplicial subset consisting of natural transformations of simplicial functors $(\text{Art}_k^{\text{skel}} \wr F) \rightarrow s\text{Sets}$ from the simplicial functor $N(- \downarrow (\text{Art}_k^{\text{skel}} \wr F)^{\text{op}})$ to the functor $(S, x) \mapsto S$. This simplicial set R^F inherits a ring structure from the S : indeed R^F is the “homotopy limit” of a simplicial diagram in $\text{SCR}_{/k}$ and in fact defines an object of $\text{SCR}_{/k}$. In general we should not expect to have $R^F \in \text{Art}_k$ and R^F can be quite large without further assumptions on F . However, if F is representable we will have $R^F \in \text{Art}_k$ and in fact it will be a representing object for F . More precisely, there is a canonical natural transformation

$$M^F(A) \rightarrow \text{Hom}(R^F, A),$$

and the precise “functorial approximation” of representable functors $F : \text{Art}_k \rightarrow s\text{Sets}$ is the following result.

Proposition 2.18. *The associations $F \mapsto M^F$ and $F \mapsto R^F$ are functorial. If $F \rightarrow F'$ is a natural weak equivalence then so is $M^F \rightarrow M^{F'}$, and $R^{F'} \rightarrow R^F$ is a weak equivalence. The maps*

$$F(A) \leftarrow M^F(A) \rightarrow \text{Hom}(R^F, A) \rightarrow \text{Hom}(c(R^F), A)$$

defined above are natural in both $A \in \text{Art}_k$ and F , where $c : \text{SCR}_{/k} \rightarrow \text{SCR}_{/k}$ denotes a functorial cofibrant approximation. The functors M^F and $\text{Hom}(c(R^F), -)$ are homotopy invariant. Moreover the left hand map is a natural weak equivalence for all F (simplicially enriched, homotopy invariant, and Kan valued) and the composition $M^F(A) \rightarrow \text{Hom}(c(R^F), A)$ is a weak equivalence for representable F .

Proof sketch. It remains to see that $M^F(A) \rightarrow \text{Hom}(c(R^F), A)$ is a weak equivalence for all A when F is representable. It suffices to consider $A \in \text{Art}_k^{\text{skel}}$ and

$F = \text{Hom}(R, -)$ for $R \in \text{Art}_k^{\text{skel}}$. In this case the simplicial category $(\text{Art}_k^{\text{skel}} \wr F)$ has a terminal object given by (R, id) and hence the homotopy limit projects by a weak equivalence to the value at that object, giving a weak equivalence $R^F \rightarrow R$ and hence a natural weak equivalence $F = \text{Hom}(R, -) \rightarrow \text{Hom}(R^F, -)$. It remains to check that this natural weak equivalence commutes with the maps asserted in the statement of the proposition, which we leave to the reader. \square

2.6. Pro-representable functors. The class of pro-representable functors $\text{Art}_k \rightarrow s\text{Sets}$ can be defined in several equivalent ways. Let us pick the following as the official definition.

Definition 2.19. *A functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ is pro-representable if there exists a pro-object $R = (j \mapsto R_j)$ indexed by a filtered category J and with all $R_j \in \text{Art}_k$ cofibrant, such that \mathcal{F} is naturally weakly equivalent to the functor*

$$A \mapsto \text{colim}_{j \in J^{\text{op}}} \text{Hom}(R_j, A).$$

\mathcal{F} is sequentially pro-representable if it is possible to choose J countable. (It is well known that this implies that in fact one can choose $J = (\mathbb{N}, <)$: argue as in Lemma 2.22 below to replace J by a partially ordered set, and then the argument of [32, Tag 0597] to convert it to $(\mathbb{N}, <)$.)

Let us point out that any pro-representable functor is automatically homotopy invariant, since, as we already observed, $\text{Hom}(R_j, -)$ is homotopy invariant for cofibrant R_j , and since filtered colimit of commutes with homotopy groups. Following Lurie and Schlessinger, we shall discuss general criteria for pro-representability. First let us discuss some more easily deduced reformulations.

Lemma 2.20. *Let $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ be a simplicially enriched functor with values in Kan simplicial sets. Then the following conditions are equivalent.*

- (i) \mathcal{F} is pro-representable.
- (ii) There exists a natural weak equivalence

$$\text{hocolim}_{j \in J^{\text{op}}} \text{Hom}(R_j, -) \rightarrow \mathcal{F}$$

with $R = (j \mapsto R_j)$ as above.

- (iii) There exists a filtered category J and a functor $F : J^{\text{op}} \times \text{Art}_k \rightarrow s\text{Sets}$ with $F(j, -)$ representable (in the sense of Definition 2.14) for all $j \in J$ and such that \mathcal{F} is naturally weakly equivalent to $A \mapsto \text{colim} F(j, A)$

Proof. That (ii) implies (i) follows from the fact that $\text{hocolim} \rightarrow \text{colim}$ is a weak equivalence for filtered indexing categories. That (i) implies (iii) is clear, so it remains to see that (iii) implies (ii).

Applying the functorial procedure from Proposition 2.18 to each $F(j, -)$ we obtain cofibrant $R_j \in \text{Art}_k$ and zig-zags of weak equivalences

$$\text{Hom}(R_j, -) \leftarrow M_j \rightarrow F(j, -),$$

all natural with respect to $j \in J$. To split the map $M_j \rightarrow \text{Hom}(R_j, -)$ up to homotopy it suffices to pick a lift of the identity under $M_j(R_j) \rightarrow \text{Hom}(R_j, R_j)$, which

is possible, but a bit of care is needed to assemble these to a natural transformation $\mathrm{hocolim}_j \mathrm{Hom}(R_j, -) \rightarrow M_j$. This follows from Lemma 2.21 below, which implies that any (simplicially enriched, Kan valued, homotopy invariant) functor \mathcal{F} connected to $\mathrm{hocolim}_j \mathrm{Hom}(R_j, -)$ by a zig-zag of natural weak equivalences in fact admit a single natural weak equivalence $\mathrm{hocolim}_j \mathrm{Hom}(R_j, -)$. \square

Lemma 2.21. *Let J be a filtered category and $R : J^{\mathrm{op}} \rightarrow \mathrm{Art}_k$ a pro-object with $R_j \in \mathrm{Art}_k$ cofibrant for all $j \in J$. Then for any diagram of (simplicially enriched, Kan valued, homotopy invariant) functors $\mathrm{Art}_k \rightarrow s\mathrm{Sets}$ and natural transformations*

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \simeq \\ \mathrm{hocolim}_{j \in J} \mathrm{Hom}(R_j, -) & \longrightarrow & \mathcal{G} \end{array}$$

there exists a natural transformation $\mathrm{hocolim}_j \mathrm{Hom}(R_j, -) \rightarrow \mathcal{F}$ and a natural simplicial homotopy making the diagram homotopy commute.

Proof. The universal property of $\mathrm{hocolim}$ implies that the set of natural transformations $\mathrm{hocolim}_j \mathrm{Hom}(R_j, -) \rightarrow \mathcal{G}$ is in bijection with vertices in $\mathrm{holim}_j \mathcal{G}(R_j)$ and that natural simplicial homotopies are in bijection with 1-simplices in this homotopy limit. Hence the claim follows from the weak equivalence

$$\mathrm{holim}_j \mathcal{F}(R_j) \rightarrow \mathrm{holim}_j \mathcal{G}(R_j). \quad \square$$

If $(i \mapsto R_i)$ is an object of $\mathrm{pro}\text{-}\mathrm{Art}_k$ with all R_i cofibrant, then the natural map $\mathrm{hocolim}_i \mathrm{Hom}(R_i, -) \rightarrow \mathrm{colim}_i \mathrm{Hom}(R_i, -)$ is an objectwise weak equivalence so from a homotopical point of view the two functors should be considered interchangeable. Nevertheless the homotopy colimit and colimit have different technical properties, in particular the homotopy colimit is “easier to map out of”, cf. Lemma 2.21, and the strict colimit has the convenient property that it sends fibrations in Art_k to Kan fibrations of simplicial sets. In Lemma 2.24 below, we shall establish a criterion for factoring a natural transformation out of $\mathrm{hocolim}_i \mathrm{Hom}(R_i, -)$ through a natural transformation out of the strict colimit.

Lemma 2.22. *Any pro-object of Art_k is isomorphic to an $R : I^{\mathrm{op}} \rightarrow \mathrm{Art}_k$ whose indexing category is a directed set, equipped with a strictly increasing map $I \rightarrow \mathbb{N}$, and satisfying that $\{j \in I \mid j \leq i\}$ is finite for all $i \in I$. For any such diagram R there exists another diagram $R' : I^{\mathrm{op}} \rightarrow \mathrm{Art}_k$ (with the same indexing category, if desired) and a natural transformation $R \rightarrow R'$ such that each $R_i \rightarrow R'_i$ is a weak equivalence and also a cofibration (in particular R'_i is cofibrant if R_i is), and such that the natural map*

$$R'_i \rightarrow \lim_{j < i} R'_j$$

is a fibration.

Proof. The indexing category map be replaced in the usual way (cf. [1, Expose 1, Prop 8.1.6]): if I does not already satisfy this, let J be the poset of finite subdiagrams of I having a unique terminal object, ordered by inclusion (where a “diagram” is a set O of objects and a set F of morphisms such that the source and target of any element of F are in O , and “terminal object” means an object $x \in O$ such that any object $y \in O$ admits precisely one morphism $y \rightarrow x$ in F). Then J is a directed set and $J \rightarrow \mathbb{N}$ sends a diagram to its cardinality. The functor $J \rightarrow I$, which sends a subdiagram to its terminal object, is cofinal and hence the original pro-object $R : I^{\text{op}} \rightarrow \text{Art}_k$ is isomorphic to the composition with $J \rightarrow I$.

Now suppose I satisfies the assumption and $R : I^{\text{op}} \rightarrow \text{Art}_k$. Then we construct R' by the same argument as when constructing the “Reedy model structure” on I -shaped diagrams. Namely, filter I by the full subcategories $I_{\leq n}$ on those objects with image in $\{0, \dots, n\} \subset \mathbb{N}$, and assume the restriction $R' : I_{\leq n-1}^{\text{op}} \rightarrow \text{Art}_k$ and the maps $R_i \rightarrow R'_i$ have been defined for $i \in I_{\leq n-1}$. Then for $i \in \text{Ob}(I_{\leq n}) \setminus \text{Ob}(I_{\leq n-1})$ define $R_i \rightarrow R'_i$ by factoring the composition

$$R_i \rightarrow \lim_{j < i} R_j \rightarrow \lim_{j < i} R'_j$$

into an acyclic cofibration $R_i \rightarrow R'_i$ followed by a fibration $R'_i \rightarrow \lim_{j < i} R'_j$. This is possible in $\text{SCR}_{/k}$ and the result will be in Art_k because R_i is. The resulting object comes with a map to $\lim_{j < i} R'_j$ which, as i varies, precisely contains the information to make $i \mapsto R'_i$ into a functor $I_{\leq n}^{\text{op}} \rightarrow \text{Art}_k$ extending the previously constructed functor on $I_{\leq n-1}$. \square

The conclusion of the Lemma above motivates the following definition, which is a generalization of “tower of fibrations between cofibrant objects”.

Definition 2.23. *Let us say that a filtered diagram $R : I^{\text{op}} \rightarrow \text{Art}_k$ is nice if I is a poset such that $\{j \in I \mid j \leq i\}$ is finite for all i , and there exists a strictly increasing poset map $I \rightarrow \mathbb{N}$, that each $R_i \in \text{Art}_k$ is cofibrant, and that each of the maps $R_i \rightarrow \lim_{j < i} R_j$ is a fibration.*

By the Lemma above we may replace any pro-object $R : I^{\text{op}} \rightarrow \text{Art}_k$ by one given by a nice diagram $R' : J \rightarrow \text{Art}_k$. In general this replacement is a zig-zag $R \leftarrow R'' \rightarrow R'$ of morphisms in pro-Art_k , the arrow $R'' \rightarrow R$ changing I to J and cofibrantly replacing all R_i , the arrow $R'' \rightarrow R'$ as in the lemma.

Lemma 2.24. *Let $R : I \rightarrow \text{Art}_k$ be a nice pro-object, and let $\mathcal{F} : \text{SCR}_{/k} \rightarrow s\text{Sets}$ be a simplicially enriched functor which (strictly) preserves finite limits and takes fibrations in Art_k to Kan fibrations. Then any natural transformation*

$$T : \text{hocolim}_{i \in I} \text{Hom}(R_i, -) \rightarrow \mathcal{F}$$

is naturally homotopic to one which factors through a natural transformation out of the (strict) colimit.

Before giving the proof, let us mention that for us the main example of a functor satisfying the assumption in the Lemma is $\mathcal{F} = \text{colim}_{j \in J} \text{Hom}(R'_j, -)$ for $R' \in$

pro-Art_k with all R'_j cofibrant. In this case we conclude that (when the individual R'_j are cofibrant and the pro-object R is nice in the sense defined above) any natural transformation

$$\text{hocolim}_{i \in I} \text{Hom}(R_i, -) \rightarrow \text{colim}_{j \in J} \text{Hom}(R'_j, -)$$

is naturally homotopic to a transformation which factors through the colimit, and hence is induced by a morphism $R' \rightarrow R$ in pro-Art_k .

Proof. By the enriched Yoneda lemma, T corresponds (bijectively) to a vertex in $\text{holim}_{i \in I} \mathcal{F}(R_i)$, and we want a 1-simplex connecting T to a vertex in $\lim_{i \in I} \mathcal{F}(R_i) \subset \text{holim}_{i \in I} \mathcal{F}(R_i)$. Now the assumptions imply that all the maps

$$\mathcal{F}(R_i) \rightarrow \lim_{j < i} \mathcal{F}(R_j) = \mathcal{F}(\lim_{j < i} R_j)$$

are fibrations. This condition in turn implies that $\lim_{i \in I} \mathcal{F}(R_i) \rightarrow \text{holim}_{i \in I} \mathcal{F}(R_i)$ is a weak equivalence (indeed, $\lim_{i \in I \leq n} \mathcal{F}(R_i) \rightarrow \lim_{i \in I \leq n-1} \mathcal{F}(R_i)$ is a Kan fibration and $\lim_{i \in I \leq n} \mathcal{F}(R_i) \rightarrow \text{holim}_{i \in I \leq n} \mathcal{F}(R_i)$ a weak equivalence, by induction on n). \square

Next we study natural transformations between pro-representable functors. As in the representable case, natural transformations correspond to morphisms between the pro-objects representing them, but the precise statement is a bit more complicated in this case.

Lemma 2.25. *Let \mathcal{F} and \mathcal{G} be pro-representable with Kan values and suppose given natural weak equivalences $\text{hocolim}_i \text{Hom}(R_i, -) \rightarrow \mathcal{F}$ and $\text{hocolim}_j \text{Hom}(R'_j, -) \rightarrow \mathcal{G}$ with all R_i and R'_j cofibrant and $(i \mapsto R_i) \in \text{pro-Art}_k$ nice in the sense above.*

For any natural transformation $T : \mathcal{F} \rightarrow \mathcal{G}$ there exists a morphism $T' : R' \rightarrow R$ between the representing pro-objects, and a diagram of functors and natural transformations

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{T} & \mathcal{G} \\ \uparrow \simeq & & \uparrow \simeq \\ \text{hocolim}_j \text{Hom}(R_j, -) & \longrightarrow & \text{hocolim}_j \text{Hom}(R'_j, -) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{colim}_j \text{Hom}(R_j, -) & \xrightarrow{T'} & \text{colim}_j \text{Hom}(R'_j, -), \end{array}$$

where both squares commute up to natural simplicial homotopy.

Proof sketch. This is an easy consequence of Propositions 2.21 and 2.24. \square

Although we shall not explicitly use it, we remark that there is a model structure on the category of simplicially enriched functors $\text{Art}_k \rightarrow s\text{Sets}$ in which the fibrations and weak equivalences are defined objectwise, cf. e.g. [36]. In particular there is a well defined homotopy category of such functors. The above Lemma then

implies that T and $\operatorname{colim}_j \operatorname{Hom}(R_j, -) \rightarrow \operatorname{colim}_j \operatorname{Hom}(R'_j, -)$ are isomorphic in the category of arrows in the homotopy category of functors.

Remark. By general theory, the bottom natural transformation corresponds to a morphism from the object of $\operatorname{pro}\text{-}\operatorname{Art}_k$ given by the R'_j to the object given by the R_j . Similarly, the natural transformation T' corresponds to a zero-simplex in $\operatorname{holim}_j \operatorname{hocolim}_i \operatorname{Hom}(R'_i, R_j)$.

3. MORE ON REPRESENTABLE FUNCTORS

We continue our study of representable functors on the category Art_k .

3.1. Postnikov truncations. A basic construction in homotopy theory associates functorially to any topological space X a space $\tau_{\leq n} X$ with vanishing homotopy groups in dimensions above n and an $(n+1)$ -connected map $X \rightarrow \tau_{\leq n} X$. In fact, the map $X \rightarrow \tau_{\leq n} X$ is unique up to weak equivalence in an appropriate sense. In topological spaces the usual proof constructs $\tau_{\leq n}$ from X by attaching $(n+2)$ -cells along *all possible* maps $\partial D^{n+2} \rightarrow X$, then $(n+3)$ -cells to the result along all possible maps from ∂D^{n+3} , etc.

The same idea as in topological spaces may be used to construct a functor $\tau_{\leq n} : \operatorname{SCR} \rightarrow \operatorname{SCR}$ and a natural map $R \rightarrow \tau_{\leq n} R$ in SCR , inducing an isomorphism in π_i for $i \leq n$ and such that $\pi_i \tau_{\leq n} R = 0$ for $i > n$. For technical reasons it is convenient to have $R \rightarrow \tau_{\leq n} R$ always be a cofibration, such that e.g. $\tau_{\leq n} R$ is cofibrant when R is. One construction of such a functor proceeds as for topological spaces, using cell attachments in SCR : first attach cells along all possible maps $\partial \Delta^{n+2} \rightarrow R$, then attach cells along all possible maps from $\partial \Delta^{n+3}$ to the result, etc. Then $\tau_{\leq n} R$, defined as the union of these cell attachments, has the required homotopical properties and $R \rightarrow \tau_{\leq n} R$ is a cofibration by construction.

Alternatively we can use the *coskeleton* functors: for any simplicial set Y there is a natural transformation $Y \rightarrow \operatorname{cosk}^n(Y)$ with the universal property that maps $X \rightarrow \operatorname{cosk}^n(Y)$ are in natural bijection with maps from the n -skeleton of X to Y . When Y is Kan it is easy to see that $Y \rightarrow \operatorname{cosk}^n(Y)$ is a model for $Y \rightarrow \tau_{\leq n} Y$. In particular, since the underlying simplicial set of any $R \in \operatorname{SCR}$ is automatically Kan, we could alternatively define $\tau_{\leq n} R$ by a (functorial) factorization $R \rightarrow \tau_{\leq n} R \rightarrow \operatorname{cosk}^n(R)$ into a cofibration followed by an acyclic fibration. This construction has the mild advantage that as n varies the $\tau_{\leq n} R$ fit into a tower $\cdots \rightarrow \tau_{\leq n+1} R \rightarrow \tau_{\leq n} R \rightarrow \cdots \rightarrow \tau_{\leq 0} R$. We will use this as the “official” definition of truncation, although it is slightly less intuitive than the previous one.

If $R \in \operatorname{SCR}_{/k}$, then $\tau_{\leq n} R$ also comes with a natural morphism to k so (at least for $n \geq 0$) we may also regard $\tau_{\leq n}$ as a functor $\operatorname{SCR}_{/k} \rightarrow \operatorname{SCR}_{/k}$. If $R \in \operatorname{Art}_k$ then also $\tau_{\leq n} R \in \operatorname{Art}_k$.

3.2. Tensor product of simplicial rings and pro-simplicial rings. Recall that if $R' \leftarrow R \rightarrow R''$ is any pushout diagram in SCR we defined $R' \otimes_R R''$ as the tensor product applied in each degree. This tensor product is the (strict) pushout in

simplicial commutative rings, and hence for any simplicial ring A , the diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Hom}(R' \otimes_R R'', A) & \longrightarrow & \mathrm{Hom}(R', A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(R'', A) & \longrightarrow & \mathrm{Hom}(R, A) \end{array}$$

is pullback (not necessarily homotopy pullback). Indeed, since A may be replaced by $A^{\Delta[p]}$ it suffices to prove this on zero-simplices of mapping spaces, where it follows by applying the analogous result for discrete rings in each degree. If either $R \rightarrow R'$ or $R \rightarrow R''$ is a cofibration in simplicial commutative rings, the square above is also homotopy cartesian.

We prove the following homotopical analogue.

Proposition 3.1. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_{01}$ be representable functors $\mathrm{Art}_k \rightarrow s\mathrm{Sets}$ and let $T_i : \mathcal{F}_i \rightarrow \mathcal{F}_{01}$ be natural transformations, $i = 0, 1$. Then the objectwise homotopy pullback*

$$A \mapsto \mathcal{F}_0(A) \times_{\mathcal{F}_{01}(A)}^h \mathcal{F}_1(A)$$

is sequentially pro-representable.

Proof sketch. By homotopy invariance of the homotopy pullback, we may replace \mathcal{F}_i and \mathcal{F}_{01} by naturally weakly equivalent functors and also the natural transformations with naturally homotopic ones.

Suppose $R_i, R_{01} \in \mathrm{Art}_k$ are cofibrant and $\iota_i \in \mathcal{F}(R_i)$ and $\iota_{01} \in \mathcal{F}(R_{01})$ induce natural weak equivalences, and suppose $R_{01} \rightarrow R_i$ represent the natural transformations $\mathcal{F}_i \rightarrow \mathcal{F}_{01}$, as in Lemma 2.25. We may suppose that $R_{01} \rightarrow R_i$ are cofibrations, in which case $\mathrm{Hom}(R_i, A) \rightarrow \mathrm{Hom}(R_{01}, A)$ are fibrations, so the objectwise strict pullback

$$A \mapsto \mathrm{Hom}(R_0, A) \times_{\mathrm{Hom}(R_{01}, A)} \mathrm{Hom}(R_1, A)$$

is naturally weakly equivalent to the homotopy pullback $\mathcal{F}_0 \times_{\mathcal{F}_{01}}^h \mathcal{F}_1$. This objectwise fiber product is naturally isomorphic to the functor

$$A \mapsto \mathrm{Hom}(R_0 \otimes_{R_{01}} R_1, A),$$

and $R_0 \otimes_{R_{01}} R_1$ is cofibrant. Unfortunately $R_0 \otimes_{R_{01}} R_1$ need not be an object of Art_k , but it is close. Clearly the discrete ring

$$\pi_0(R_0 \otimes_{R_{01}} R_1) = \pi_0(R_0) \otimes_{\pi_0(R_{01})} \pi_0(R_1)$$

is Artin local in the usual sense, and the spectral sequence [28, Theorem 6, §6, II]

$$E^2 = \mathrm{Tor}_{\pi_*(R_{01})}(\pi_*(R_0), \pi_*(R_1)) \Rightarrow \pi_*(R_0 \otimes_{R_{01}} R_1),$$

shows that $\pi_k(R_0 \otimes_{R_{01}} R_1)$ has finite length as a module over π_0 for each $k > 0$, but it could be non-zero for infinitely many k . However, any map

$$R_0 \otimes_{R_{01}} R_1 \rightarrow A$$

with $A \in \text{Art}_k$ admits a factorization over some finite Postnikov truncation $\tau_{\leq n}(R_0 \otimes_{R_{01}} R_1)$, so the inverse system given by (cofibrant approximations to) $\tau_{\leq n}(R_0 \otimes_{R_{01}} R_1)$ will pro-represent the functor, and each truncation is Artin. \square

The same statement holds for pro-representable functors:

Proposition 3.2. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_{01}$ be pro-representable functors $\text{Art}_k \rightarrow \text{sSets}$ and let $T_i : \mathcal{F}_i \rightarrow \mathcal{F}_{01}$ be natural transformations, $i = 0, 1$. Then the homotopy pullback*

$$A \mapsto \mathcal{F}_0(A) \times_{\mathcal{F}_{01}(A)}^h \mathcal{F}_1(A)$$

is also pro-representable.

If all three functors are sequentially pro-representable, then so is the pullback.

Proof. As before we may use homotopy invariance to replace \mathcal{F}_i by a functor $F_i = \text{colim}_j \text{Hom}(R_i^j, -)$ for $(j \mapsto R_i^j) \in \text{pro-Art}_k$ and similarly for \mathcal{F}_{01} . We may also assume the representing pro-objects are *nice*, and then replace (by Lemma 2.25) the natural transformations by those induced by morphisms

$$\phi_i \in \lim_k \text{colim}_j \text{Hom}(R_{01}^j, R_i^k)$$

in the pro-category.

Now arrange that $\text{colim}_j \text{Hom}(R_{01}^j, A) \rightarrow \text{colim}_j \text{Hom}(R_1^j, A)$ is a fibration for all A , for example in the following way. After replacing with isomorphic objects, we may assume that they have a common indexing category J , and that the morphism in the pro-category is given by a natural transformation of functors $R_1 \Rightarrow R_{01} : J^{\text{op}} \rightarrow \text{Art}_k$, and then functorially replacing each constituent morphism $R_1^j \rightarrow R_{01}^j$ by a cofibration. (The pro-objects may not stay nice, but that is no longer important.) Since filtered colimits preserve Kan fibrations, the resulting map of simplicial sets $\text{colim}_j \text{Hom}(R_{01}^j, A) \rightarrow \text{colim}_j \text{Hom}(R_1^j, A)$ is a Kan fibration for all A . After this replacement the pullback is weakly equivalent to the homotopy pullback, and we see that it is “represented” by the object $(j \mapsto R_1^j \otimes_{R_{01}^j} R_0^j) \in \text{pro-SCR}_{/k}$. As in Proposition 3.1, the levels of this need not be objects of Art_k , but we can then instead consider the pro-object

$$c \left(\tau_{\leq n} (R_1^j \otimes_{R_{01}^j} R_0^j) \right)_{(j,n) \in J \times \mathbb{N}},$$

where $c(-)$ denotes a choice of functorial cofibrant replacement. \square

Inspired by this, we will use the following *ad hoc* version of a derived tensor product in pro-Art_k . The definition depends on a choice of functorial factorization of morphisms in $\text{SCR}_{/k}$ (or just Art_k) into a cofibration followed by an acyclic fibration, as well as a sequence of Postnikov truncation functors $\tau_{\leq n}$.

Definition 3.3. *Given morphisms $\phi : R \rightarrow R'$ and $R \rightarrow R''$ in pro-Art_k , represent them by natural transformations $(R_j \rightarrow R'_j)_{j \in J}$ and $(R_j \rightarrow R''_j)_{j \in J}$ of pro-objects indexed by the same category, apply functorial factorization to replace these maps*

by $R_j \rightarrow \widehat{R'_j} \xrightarrow{\sim} R'_j$ and $R_j \rightarrow \widehat{R''_j} \xrightarrow{\sim} R''_j$ (where the first map of each is a cofibration, and the second a weak equivalence) and define

$$R' \underline{\otimes}_R R'' = \text{nice } c \left(\tau_{\leq n}(\widehat{R'_j} \otimes_{R_j} \widehat{R''_j}) \right)_{(j,n) \in J \times \mathbb{N}},$$

where as usual c denotes the chosen cofibrant replacement functor, and nice refers to the procedure from Lemma 2.22, replacing a pro-object by an equivalent nice one.

By our discussion above, $R' \underline{\otimes}_R R''$ pro-represents the functor

$$\text{colimHom}(R'_j, -) \times_{\text{colimHom}(R_j, -)}^h \text{colimHom}(R''_j, -).$$

Let us point out that the definition above is a bit of a kludge, and the notation $R' \underline{\otimes}_R R''$ is shorthand for a construction whose isomorphism class depends on many choices. In particular, it does not define a functor from pushout diagrams in pro-Art_k , because the choice of how to represent $R'' \leftarrow R \rightarrow R'$ by natural transformations of functors out of the same indexing category is not functorial. Different choices lead to non-isomorphic pro-objects $R' \underline{\otimes}_R R''$, but all of them are nice and represent the same functor so they are at least related by morphisms in pro-Art_k inducing objectwise weak equivalences of represented functors.

3.3. Some pro-representable functors. We discuss a few examples of functors for which we may describe explicit pro-representing objects.

Proposition 3.4. *The terminal functor, defined as $\mathcal{F}(A) = \{*\}$, is sequentially pro-representable.*

Proof. Let $p = \text{char}(k)$, let $F(n, A)$ be the subspace of $\text{Hom}(\Delta[1], A)$ consisting of 1-simplices starting at 0 and ending at $p^n 1$, and define $F(n, A) \rightarrow F(n+1, A)$ as $x \mapsto px$. Since $\pi_0(A)$ is Artinian, $F(n, A)$ is non-empty for large enough n , in which case it has the homotopy type of the loop space of $(A, 0)$. More explicitly, if we pick $s_n \in F(n, A)$ and define $s_{n+k} = p^k s_n \in F(n+k, A)$, we may define an isomorphism of simplicial sets $\Omega(A, 0) \rightarrow F(n+k, A)$ by $x \mapsto x + s_{n+k}$. With respect to these isomorphisms, the map $F(n, A) \rightarrow F(n+1, A)$ is identified with simplicial loops of multiplication by $p : A \rightarrow A$ and hence induces multiplication by p on homotopy groups. Since the homotopy groups are finite and hence p -torsion, $\text{colim}_n \pi_k(F(n, A)) = 0$ and hence $\text{colim}_n F(n, A)$ is contractible for all A .

Finally we have a natural weak equivalence $F(n, A) \simeq \text{Hom}(R_n, A)$, where R_n is a cofibrant approximation to $W(k)/p^n$ – e.g., if $k = \mathbb{F}_p$, we may take R_n to be the ring obtained by freely adjoining to the discrete simplicial ring \mathbb{Z} a generator y_1 satisfying $d_0 y_1 = 0, d_1 y_1 = p^n$. \square

We shall sometimes simply write $W(k)$ for a pro-representing object of the terminal functor, but this is somewhat sloppy. The constant object of SCR_k given by $W(k)$ is not cofibrant and neither is the constant object $W(k)/p^n$.

Remark. It is easy to see that the terminal functor is only pro-representable, not representable. Indeed, any putative representing object R would have $p^n = 0 \in \pi_0(R)$ for some n and hence have $\text{Hom}(R, A) = \emptyset \neq *$ when $\pi_0(A) = W(k)/p^{n+1}$.

Lemma 3.5. *The functor $A \mapsto \mathfrak{m}(A) = \text{Ker}(A \rightarrow k)$ is sequentially pro-representable.*

Proof. Following the same strategy as in the previous lemma, we let $F(n, A)$ be the subspace of $\mathfrak{m} \times A^{\Delta[1]} \times A^{\Delta[1]}$ consisting of triples (a, λ, σ) where λ is a path from $p^n 1$ to 0 as before and σ is a path from a^n to 0 . There is a natural transformation $F(n, A) \rightarrow \mathfrak{m}$ sending (a, λ, σ) to a and a compatible one $F(n, A) \rightarrow F(n+1, A)$ sending $(a, \lambda, \sigma) \mapsto (a, p\lambda, a\sigma)$, inducing a weak equivalence $\text{colim}_n F(n, A) \rightarrow \mathfrak{m}(A)$. Each $F(n, -)$ is representable by a cofibrant approximation to $W(k)[x]/(x^n, p^n)$ and hence \mathfrak{m} is sequentially pro-representable. \square

Lemma 3.6. *For $n \geq 0$ the functor $A \mapsto \Omega^n \mathfrak{m}(A)$ is sequentially pro-representable.*

Proof. By Proposition 3.2 this follows inductively by writing $\Omega^n \mathfrak{m}(A)$ as the homotopy pullback of $* \rightarrow \Omega^{n-1} \mathfrak{m}(A) \leftarrow *$. \square

Remark. A representing pro-object in the above Lemma can be constructed more explicitly using the “cell attachment” construction from section 2.1. In fact, if $n > 0$ and $\mathbb{Z}[y]$ is the simplicial commutative ring obtained by adjoining a (trivially attached) n -cell to the initial object \mathbb{Z} , i.e. the level-wise free commutative algebra on the pointed simplicial set S^n , then this object “represents” in the sense that it is cofibrant in SCR and $\text{SCR}(\mathbb{Z}[y], A)$ is naturally isomorphic to $\Omega^n A = \Omega^n \mathfrak{m}(A)$. Of course $\mathbb{Z}[y] \rightarrow k$ is not an object of Art_k because the homotopy groups of $\mathbb{Z}[y]$ are too large, and in fact the functor is not representable. But if we write $W = W(k)$ we may instead use the pro-objects formed by cofibrant approximations to the truncations $\tau_{\leq n}(W/p^n)[y]$, $n \in \mathbb{N}$. If as usual we write c for a chosen cofibrant replacement functor, we shall occasionally denote the resulting pro-object $(n \mapsto c(\tau_{\leq n}(W/p^n)[y]))$ informally by $W[[y]]$, thinking of it as power series over W in one variable y of degree n .

More generally we have the following definition

Definition 3.7. *Suppose $\mathcal{F} : \text{Art}_k \rightarrow \text{sSets}$ is a homotopy invariant functor and $e : \mathcal{F}(A) \rightarrow \Omega^n \mathfrak{m}(A)$ is a natural transformation. We may then define a functor $\mathcal{F}' : \text{Art}_k \rightarrow \text{sSets}$ by letting $\mathcal{F}'(A)$ to be the homotopy fiber of $e : \mathcal{F}(A) \rightarrow \Omega^n \mathfrak{m}(A)$ over the basepoint 0 , i.e. \mathcal{F}' is defined by the following homotopy pullback square:*

$$(3.1) \quad \begin{array}{ccc} \mathcal{F}'(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{e} & \Omega^n \mathfrak{m}(A). \end{array}$$

In this case we shall say that \mathcal{F}' is obtained from \mathcal{F} by attaching an $(n+1)$ -cell to \mathcal{F} along e . For $n = -1$ we shall simply define $\mathcal{F}'(A) = \mathcal{F} \times \mathbf{m}(A)$ and say that \mathcal{F}' is obtained from \mathcal{F} by attaching a 0-cell.

If $e : \mathcal{F} \rightarrow \Omega^n \mathbf{m}$ is a natural transformation and \mathcal{F}' is defined as above, then if \mathcal{F} is (pro-)representable by $R \in \text{pro-Art}_k$, the functor \mathcal{F}' is (pro-)representable by the derived tensor product $W \underline{\otimes}_{W[[x]]} R$, where $W[[x]] \rightarrow R$ is a morphism in pro-Art_k classifying $[e] \in \pi_n \mathbf{m}(R)$.

3.4. Formally cohesive functors. In section 4.6 below, we shall review a verifiable criterion for a homotopy invariant functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ to be pro-representable. The criterion, due to Lurie, is a simplicial version of Schlessinger's criterion. The criterion makes two assumptions on \mathcal{F} , one of which we discuss in this section.

Any functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ will send a commutative square

$$(3.2) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in Art_k to a commutative square

$$(3.3) \quad \begin{array}{ccc} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(B) \\ \downarrow & & \downarrow \\ \mathcal{F}(C) & \longrightarrow & \mathcal{F}(D) \end{array}$$

of simplicial sets, by definition of “functor”. Recall that a strictly commutative square is said to be *homotopy cartesian* if the induced map from the initial vertex to the homotopy pullback of the rest of the diagram is a weak equivalence. We shall be particularly interested in homotopy invariant functors satisfying the following properties.

Definition 3.8. *Let $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ be a homotopy invariant functor.*

- (i) \mathcal{F} preserves homotopy pullback if the square (3.3) is homotopy cartesian whenever the diagram (3.2) is (strictly) cartesian and $C \rightarrow D$ and $B \rightarrow D$ are surjective in each simplicial degree.
- (ii) [Lurie] \mathcal{F} is formally cohesive (or “good”) if it preserves homotopy pullback and if $\mathcal{F}(k)$ is contractible.

Let us point out that a map of simplicial rings is surjective in all simplicial degrees if and only if it is a fibration and induces a surjection in π_0 .

Lemma 3.9. *If a homotopy invariant functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ preserves homotopy pullback in the sense above, it also preserves homotopy pullback in the stronger sense that the square (3.3) is homotopy cartesian whenever the square (3.2) is (strictly) cartesian and either $C \rightarrow D$ or $B \rightarrow D$ are surjections in all simplicial degrees.*

Proof. See Lemma 6.2.7 in Lurie's thesis [22]. \square

Example 3.10. If $R \in \mathbf{Art}_k$ is any object, the functor $\mathcal{F}_R = \mathrm{Hom}(R, -)$ may not be homotopy invariant, but if R is cofibrant it will be (Lemma 2.15). In that case the functor \mathcal{F}_R will preserve homotopy pullback (because it preserves actual pullbacks and also Kan fibrations) but will not necessarily have $\mathcal{F}_R(k)$ contractible.

If $\bar{\rho} : R \rightarrow k$ is any zero-simplex of $\mathcal{F}_R(k)$, we may obtain a formally cohesive functor $\mathcal{F}_{R, \bar{\rho}}$ which takes $\epsilon : A \rightarrow k$ to the homotopy fiber of $\mathrm{Hom}(R, A) \rightarrow \mathrm{Hom}(R, k)$ over $\bar{\rho}$.

More generally, if $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ is homotopy invariant and preserves homotopy pullbacks and $\bar{\rho} \in \mathcal{F}(k)$ is a zero-simplex, then the functor $\mathcal{F}_{\bar{\rho}}$ which takes $(A \rightarrow k) \in \mathbf{Art}_k$ to the homotopy fiber of $\mathcal{F}(A) \rightarrow \mathcal{F}(k)$ over $\bar{\rho}$ is formally cohesive.

For formally cohesive functors we may check whether a natural transformation $T : \mathcal{F} \rightarrow \mathcal{G}$ is a natural weak equivalence without checking on all A .

Lemma 3.11. Let $T : \mathcal{F} \rightarrow \mathcal{G}$ be a natural transformation between formally cohesive functors. Then T is a natural weak equivalence if and only if $T : \mathcal{F}(k \oplus k[n]) \rightarrow \mathcal{G}(k \oplus k[n])$ is a weak equivalence for large n .

Proof. By the pullback

$$\begin{array}{ccc} k \oplus k[n] & \longrightarrow & k \oplus \widetilde{k[n+1]} \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n+1] \end{array}$$

we see that T induces an equivalence on $A = k \oplus k[n]$ for all $n \geq 0$, provided it does so for large n . The case of general A then follows from Lemma 2.8. \square

3.5. Homotopy categories. To conclude this section, let us discuss the *homotopy category* of \mathbf{Art}_k and its relation to functors $\mathbf{Art}_k \rightarrow s\mathbf{Sets}$. This discussion is not strictly necessary, but it may be helpful to orient the reader.

The homotopy category is the (non-simplicial) category $\mathrm{Ho}(\mathbf{Art}_k)$ whose objects are the objects of \mathbf{Art}_k , but whose morphism sets are given by

$$\mathrm{Ho}(\mathbf{Art}_k)(A, B) = \pi_0 \mathbf{Art}_k(c(A), B) = \pi_0 \mathbf{Art}_k(c(A), c(B))$$

where $c : \mathbf{SCR}_k \rightarrow \mathbf{SCR}_k$ is some choice of cofibrant approximation. Up to canonical isomorphism of categories, $\mathrm{Ho}(\mathbf{Art}_k)$ does not depend on the choice of c . The canonical functor $\mathbf{Art}_k \rightarrow \mathrm{Ho}(\mathbf{Art}_k)$ sends weak equivalences to isomorphisms and is universal with that property. It also has the property that two objects $A, B \in \mathbf{Art}_k$ are weakly equivalent (i.e. there is a zig-zag of weak equivalences connecting them) if and only if their image in $\mathrm{Ho}(\mathbf{Art}_k)$ are isomorphic.

For any homotopy invariant functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$, there is an associated functor

$$\begin{aligned} \mathrm{Ho}(\mathbf{Art}_k) &\rightarrow \mathbf{Sets} \\ A &\mapsto \pi_0(\mathcal{F}(A)), \end{aligned}$$

which we shall denote $\pi_0\mathcal{F}$. Of course the passage from \mathcal{F} to $\pi_0\mathcal{F}$ loses much information in general, but for formally cohesive functors we have the following result.

Lemma 3.12. *Let $\mathcal{F} \rightarrow \mathcal{G}$ be a natural transformation of formally cohesive functors $\mathbf{Art}_k \rightarrow s\mathbf{Sets}$. Assume that $\pi_0\mathcal{F}(A) \rightarrow \pi_0\mathcal{G}(A)$ is a bijection for all $A \in \mathbf{Art}_k$. Then $\mathcal{F}(A) \rightarrow \mathcal{G}(A)$ is a weak equivalence for all $A \in \mathbf{Art}_k$.*

Proof. From the homotopy pullback square in the proof of the previous lemma we obtain a natural weak equivalence $\mathcal{F}(k \oplus k[n]) \simeq \Omega\mathcal{F}(k \oplus k[n+1])$. Hence $\pi_i\mathcal{F}(k \oplus k[n]) = \pi_0\mathcal{F}(k \oplus k[n+i])$, and similarly for \mathcal{G} . It follows that $T : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ induces an isomorphism in all homotopy groups for $A = k \oplus k[n]$ for all n , and hence by the previous Lemma for all $A \in \mathbf{Art}_k$. \square

Corollary 3.13. *A formally cohesive functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ is representable if and only if $\pi_0\mathcal{F} : \mathrm{Ho}(\mathbf{Art}_k) \rightarrow \mathbf{Sets}$ is representable.*

Proof. Let \mathcal{F} be a formally cohesive functor and suppose $\pi_0\mathcal{F}$ is representable. Without loss of generality we may assume \mathcal{F} is simplicially enriched. Since $\pi_0\mathcal{F}$ is representable we may pick an object $R \in \mathbf{Art}_k$ and $\iota_0 \in \pi_0\mathcal{F}(R) = \pi_0\mathcal{F}(c(R))$ such that the corresponding natural transformation $\mathrm{Ho}(\mathbf{Art}_k)(R, -) \rightarrow \pi_0\mathcal{F}$ is a natural isomorphism. Now any choice of zero-simplex $\iota \in \mathcal{F}(c(R))$ in the path component ι_0 gives rise to a natural transformation $\mathbf{Art}_k(c(R), -) \rightarrow \mathcal{F}$ by the simplicial enrichment, and Lemma 3.12 shows that it is a natural weak equivalence.

The other direction is clear. \square

Since any representable functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ is automatically formally cohesive, we see that the question of representability splits into two: whether the functor is formally cohesive, and whether $\pi_0\mathcal{F} : \mathrm{Ho}(\mathbf{Art}_k) \rightarrow \mathbf{Sets}$ is representable in the usual sense. However, it is often *easier* to work with $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ than $\pi_0\mathcal{F} : \mathrm{Ho}(\mathbf{Art}_k) \rightarrow \mathbf{Sets}$, even when it's known a priori that \mathcal{F} is formally cohesive.

There is a parallel discussion for pro-objects and pro-representability. The functor $\mathbf{Art}_k \rightarrow \mathrm{Ho}(\mathbf{Art}_k)$ induces a functor $\mathrm{pro}\text{-}\mathbf{Art}_k$ to $\mathrm{pro}\text{-}\mathrm{Ho}(\mathbf{Art}_k)$. Of course some information is again lost in this process, but the next lemma shows that for $R \in \mathrm{pro}\text{-}\mathbf{Art}_k$ we may still recover the functor $\pi_0\mathcal{F}_R : \mathbf{Art}_k$ from the image of R in $\mathrm{pro}\text{-}\mathrm{Ho}(\mathbf{Art}_k)$.

Lemma 3.14. *Let I be a filtered category, let $R = (i \mapsto R_i) \in \mathrm{pro}\text{-}\mathbf{Art}_k$, and let $\mathcal{F}_R : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$ be the functor pro-represented by R . Then the natural maps*

$$\mathbf{Art}_k(R_i, A) \rightarrow \pi_0\mathbf{Art}_k(R_i, A) \rightarrow \mathrm{colim}_i \pi_0\mathbf{Art}_k(R_i, A) = (\mathrm{pro}\text{-}\mathrm{Ho}(\mathbf{Art}_k))(R, A)$$

induce a natural transformation

$$\pi_0 \mathcal{F}_R \rightarrow (\text{pro-Ho}(\text{Art}_k))(R, -)$$

between functors $\text{Ho}(\text{Art}_k) \rightarrow \text{Sets}$. It is a natural isomorphism if each R_i is cofibrant.

Proof. This is just the fact that π_0 takes filtered colimits (not just homotopy colimits) of simplicial sets to colimits of sets. \square

Again we advise the reader that it is better to study $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ directly than to “reduce” to $\pi_0 \mathcal{F}$. For example, it is likely easier to prove that \mathcal{F} is (pro-) representable in the homotopical sense than to prove directly that $\pi_0 \mathcal{F}$ is (pro-) representable in the ordinary categorical sense.

A useful corollary to our prior discussions is that one can meaningfully talk about the homotopy groups of a representing ring for a functor: Suppose that $R : I \rightarrow \text{Art}_k$ and $R' : J \rightarrow \text{Art}_k$ are levelwise cofibrant pro-objects of Art_k , and the functors $\mathcal{F}_R, \mathcal{F}_{R'}$ that they represent are naturally weakly equivalent. This natural weak equivalence induces, by Lemma 3.14, an equivalence $\pi_0 \mathcal{F}_R \simeq \pi_0 \mathcal{F}_{R'}$ of functors $\text{Ho}(\text{Art}_k) \rightarrow \text{Sets}$; thus we have an induced isomorphism between the images of R and R' in the pro-category of $\text{Ho}(\text{Art}_k)$. In particular, we obtain an isomorphism

$$(3.4) \quad (i \mapsto \pi_* R_i) \cong (j \mapsto \pi_* R'_j)$$

of pro-graded rings.

In the later parts of this paper, we will often use the “naive” definition

$$(3.5) \quad \pi_* R = \lim_i \pi_* R_i$$

for the homotopy groups of an object of proArt_k ; one does not expect this definition to be well-behaved in general, but in the context we will work, all the $\pi_* R_i$ are finite, and this definition has reasonable formal properties. (In general, it seems more reasonable to either work with $\pi_* R$ as a pro-object in graded rings, or to remember the topology on the limit.) The above discussion shows that, at least, $\pi_* R$ is determined, up to a unique isomorphism, by the natural weak equivalence class of \mathcal{F}_R .

4. TANGENT COMPLEXES OF RINGS AND FUNCTORS

Lurie’s *derived Schlessinger criterion* will play an important role in this paper. It is an (in practice often verifiable) criterion on a functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$, guaranteeing that it is pro-representable. We shall state it in a form suitable for our applications and outline a proof, following Lurie’s. First we must recall the *tangent complex* $\text{t}R$ of an object $R \in \text{Art}_k$ and more generally the tangent complex $\text{t}\mathcal{F}$ of a functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$.

4.1. Cohomology of (pro-) Artin rings. There are two useful ways to associate a graded k -vector space to an object $R \in \text{Art}_k$, one behaving like “cohomology” of R and one behaving like “dualized homotopy groups”. In fact, we shall define analogues of *relative* “cohomology/cohomotopy” for a morphism $R \rightarrow R'$ in Art_k .

We begin with the *relative tangent complex* which we shall study in much more detail in the following sections. Let $R \rightarrow k$ be a cofibrant object of Art_k and consider the simplicial set $\text{Art}_k(R, k \oplus k[n])$ of homomorphisms $R \rightarrow k \oplus k[n]$ lifting the given homomorphism $R \rightarrow k$. We shall be interested in the set of homotopy classes of such homomorphisms, i.e. the set $\pi_0 \text{Art}_k(R, k \oplus k[n])$. This set is canonically a k -vector space, and in fact $\text{Art}_k(R, k \oplus k[n])$ is canonically a simplicial k -vector spaces; indeed, we have a natural isomorphism of simplicial sets

$$(4.1) \quad \text{Art}_k(R, k \oplus k[n]) \cong R\text{-Mod}(\Omega_{R/\mathbb{Z}}, k[n]),$$

where $\Omega_{R/\mathbb{Z}}$ is the simplicial R -module whose p -simplices are $\Omega_{R_p/\mathbb{Z}}^1$, the Kähler differentials of $\mathbb{Z} \rightarrow R_p$, $R\text{-Mod}$ denotes the category of simplicial R -modules, simplicially enriched in the usual way, and the simplicial k -module $k[n]$ is made into a simplicial R -module using the given augmentation $R \rightarrow k$. (As usual we use the shorthand $R = (R \rightarrow k) \in \text{Art}_k$ when typographically convenient, but of course the definition depends on the homomorphism $R \rightarrow k$.)

Definition 4.1. For a cofibrant object $R \rightarrow k$ of Art_k , let us write $\pi_{-n} \mathfrak{t}R$ for the k -vector space $\pi_0 \text{Art}_k(R, k \oplus k[n])$. For a cofibration $R \rightarrow R'$ between cofibrant objects of Art_k define for $n \geq 0$ a k -modules $\pi_{-n} \mathfrak{t}(R', R)$ as π_0 of the fiber of the fibration

$$\text{Art}_k(R', k \oplus k[n]) \rightarrow \text{Art}_k(R, k \oplus k[n])$$

over the point given by the composition $R \rightarrow k \rightarrow k \oplus k[n]$. (By a similar reasoning as before, this fiber is a simplicial k -module.)

For general objects $R \in \text{Art}_k$ we define $\pi_{-n} \mathfrak{t}R$ by first taking cofibrant approximation. For a pro-object $R = (i \mapsto R_i)_{i \in I} \in \text{pro-}\text{Art}_k$ we define $\pi_{-n} \mathfrak{t}R$ as the colimit of $\pi_{-n} \mathfrak{t}R_i$. In the relative case we similarly define $\pi_{-n} \mathfrak{t}(R', R)$ for an arbitrary morphism $R \rightarrow R'$ in $\text{pro-}\text{Art}_k$.

By the discussion above the definition, the k -vector space $\pi_{-n} \mathfrak{t}R$ is identified with the André–Quillen cohomology of $\mathbb{Z} \rightarrow R$ with coefficients in the R -module k . We shall later explain in what sense $\pi_{-n} \mathfrak{t}R$ is π_{-n} of an object $\mathfrak{t}R$.

The functor $\pi_{-*} \mathfrak{t}$ can be regarded as “cohomology” of the object or pro-object R (or the “relative cohomology” of $R \rightarrow R'$) and manifestly depends only on the map of functors represented by $R \rightarrow R'$. These groups fit in a long exact sequence

$$(4.2) \quad \cdots \rightarrow \pi_{-n} \mathfrak{t}(R', R) \rightarrow \pi_{-n} \mathfrak{t}R' \rightarrow \pi_{-n} \mathfrak{t}R \rightarrow \pi_{-n-1} \mathfrak{t}(R', R) \rightarrow \cdots$$

Definition 4.2. Let $R \rightarrow R'$ be a morphism in Art_k , assume the underlying map of simplicial abelian groups is levelwise injective and let R'/R be cokernel, calculated levelwise in simplicial abelian groups. Then $R' \rightarrow R'/R$ is a Kan fibration

with fiber R , and $\pi_*(R'/R)$ is a graded module over $\pi_*(R)$ and in particular over $\pi_0(R)$. In this case, define

$$\overline{\pi}^n(R', R) = \text{Hom}_{\pi_0 R}(\pi_n(R'/R), k)$$

For a general morphism $R \rightarrow R'$ in Art_k we first replace $R \rightarrow R'$ by a cofibration of R -modules (or even just simplicial abelian groups).

For a morphism $R \rightarrow R'$ in pro-Art_k we define $\overline{\pi}^n(R', R)$ as the colimit of the level-wise $\overline{\pi}^n$.

The functor $\overline{\pi}^n$ manifestly depends only on the underlying simplicial abelian groups. While perhaps a little bit easier to define than $\pi_{-n}\mathfrak{t}R$, it seems to be less conceptually important, appearing only in a few technical proofs.

The canonical isomorphism $\pi_n(k \oplus k[n]) \rightarrow \pi_n(k \oplus k[n], k) \cong k$ gives an element $\iota \in \overline{\pi}^n(k \oplus k[n], k)$ and hence a canonical natural transformation

$$(4.3) \quad \pi_{-n}\mathfrak{t}(R', R) \rightarrow \overline{\pi}^n(R', R)$$

which could perhaps be thought of as an analogue of the homomorphism $H^n(X, Y; k) \rightarrow \text{Hom}_{\pi_1(Y, y)}(\pi_n(X, Y, y), k)$ dual to the usual Hurewicz homomorphism.

The category Art_k also has an analogue of the Hurewicz theorem.

Proposition 4.3. *Let $R \rightarrow R'$ be a morphism in pro-Art_k .*

- (i) *For $n = 0$ the homomorphism (4.3) is always injective, with image $(m/m^2)^\vee \subset m^\vee = \overline{\pi}^0(R', R)$, where $m \subset \pi_0(R' \otimes_R k)$ is the maximal ideal when $R \rightarrow R'$ is a morphism in Art_k , and m^\vee and $(m/m^2)^\vee$ are defined as the colimit when $R \rightarrow R'$ is a morphism in pro-Art_k .*
- (ii) *If $n \geq 1$ and $\overline{\pi}^l(R', R) = 0$ for $l < n$, then the homomorphism (4.3) is an isomorphism.*
- (iii) *For $n \geq 0$, we have $\overline{\pi}^l(R', R) = 0$ for all $l \leq n$ if and only if $\pi_{-l}\mathfrak{t}(R', R) = 0$ for all $l \leq n$.*

Example. Suppose that $R, R' \in \text{Art}_k$ (just for simplicity in discussing homotopy groups). If the map $\pi_{-k}\mathfrak{t}(R') \rightarrow \pi_{-k}\mathfrak{t}(R)$ is an isomorphism for $k = 0$ and an injection for $k = 1$, then

$$(4.4) \quad \pi_0 R \xrightarrow{\sim} \pi_0 R'$$

must be an isomorphism. This is a well-known statement in deformation theory, cf. e.g. [14, Theorem 2.4].

To deduce this, observe that (4.2) implies that $\pi_{-k}\mathfrak{t}(R', R)$ is vanishing for $k \leq 1$; the same conclusion then holds for $\overline{\pi}^k$. That means that in fact $\pi_k(R'/R)$ is vanishing for $k = 0, 1$, by Nakayama's lemma. Then (4.4) follows from the long exact sequence in homotopy.

Proof. Statement (i) reduces to a well known property of the map $\pi_0(R) \rightarrow \pi_0(R')$ of discrete Artin rings, and the case of pro-objects follows by taking colimit.

To prove statement (ii) we first reduce to the case $R = k$ by proving that

$$(4.5) \quad \overline{\pi}^n(R', R) \rightarrow \overline{\pi}^n(R' \otimes_R k, k)$$

is an isomorphism when $R \rightarrow R'$ is a morphism with $\overline{\pi}^l(R', R) = 0$ for $l < n$. (The corresponding result for $\mathfrak{t}(R, R')$ is straightforward.)

We first consider (4.5) in the case where R and R' are in Art_k . To this end we first reduce to the case where $k \subset \pi_0(R) = \pi_0(R')$, since both groups are unchanged by tensoring both R and R' over \mathbb{Z} with a cofibrant approximation to \mathbb{Z}/p . We have the usual spectral sequence [28, II, §6]

$$(4.6) \quad \text{Tor}_{\pi_*(R)}(\pi_*(R'/R), k) \Rightarrow \pi_*((R'/R) \otimes_R k) = \pi_*((R' \otimes_R k)/k),$$

from which it is not hard to deduce that the element of lowest degree in the target arises from $\text{Tor}_{\pi_0(R)}^0(-, k)$ applied to the lowest degree non-zero group in $\pi_*(R'/R)$. Dualizing this and using the tensor–Hom adjunction gives the claimed result (4.5).

Next we consider the case of a morphism $R \rightarrow R'$ in pro-Art_k : again we claim that if $\overline{\pi}^l(R', R) = 0$ for $l < n$, then (4.5) holds. The spectral sequence (4.6) has no direct analogue in the case of pro-objects, but it may be dualized to a more well behaved object as follows. In the case $R, R' \in \text{Art}_k$ and $\pi_0(R) = \pi_0(R') = k$, the spectral sequence (4.6) is a spectral sequence of finite dimensional k -vector spaces, and hence may be formally dualized to a spectral sequence

$$\text{Cotor}_{\pi_*(R)^\vee}(\pi_*(R'/R)^\vee, k) \Rightarrow \pi^*(R' \otimes_R k, k).$$

(The bidegrees and differentials in this spectral sequence are as in the cohomology Serre spectral sequence.) For a morphism $R \rightarrow R'$ in pro-Art_k we have such a spectral sequence levelwise, and may form the direct limit and obtain a spectral sequence, since filtered colimit is an exact functor. Here $\text{colim}_j \pi_*(R_j)^\vee$ is a coalgebra in k -vector spaces and $\text{colim}_j \pi_*(R'_j/R_j)^\vee$ is a comodule, when $\pi_0(R_j) = k$ for all j . The functor Cotor is the derived functor of cotensor product, and is calculated by a cochain complex formally dual to the “bar complex” calculating Tor . Again the lowest degree element arises from the actual cotensor product (not the higher derived ones) of $\pi_n(R'/R)^\vee$ for the smallest possible n for which that group is non-zero. In that bidegree, the Cotor is the direct limit of the k -vector spaces

$$\text{Cotor}_{\pi_0(R_j)^\vee}(\pi_n(R'_j/R_j)^\vee, k) = \text{Hom}_{\pi_0(R_j)}(\pi_n(R'_j/R_j), k),$$

and this colimit is precisely $\overline{\pi}^n(R', R)$.

We have now reduced (ii) to the case $R = k$, i.e. R' is a k -algebra. For $R' \in \text{Art}_k$ it now follows that $\tau_{\leq n} R' \simeq k \oplus V^\vee[n]$ for $V = \overline{\pi}_n(R', k) = \pi_{-n} \mathfrak{t}(R', k)$, as in the proof of Lemma 2.8.

However, the pro-case is not so easy. We shall use the truncations $\tau_{\geq n} R'$ defined for a simplicial k -algebra R' as the (homotopy) pullback of $k \rightarrow \tau_{\leq n-1} R' \leftarrow R'$.

The main ingredient in the proof of (ii) for $R = k$ for a pro-object R' , is that if $R' = (j \mapsto R'_j)_{j \in J}$, then for all $i \in J$ there exists $j > i$ such that $R'_j \rightarrow R'_i$ is homotopic to a ring map which factors through $\tau_{\geq n} R'_i \rightarrow R'_i$. This main ingredient

is proved by inductively factoring it as $R'_{j_l} \rightarrow \tau_{\geq l} R'_i \rightarrow R'_i$, with $i = j_0 < j_1 < \dots < j_n = j$. These truncations from below fit in homotopy pullback squares

$$\begin{array}{ccc} \tau_{\geq l} R'_i & \longrightarrow & k \\ \downarrow & & \downarrow \\ \tau_{\geq l-1} R'_i & \longrightarrow & k \oplus V[l-1] \end{array}$$

where $V = \pi_{l-1} R'_i$, so to lift from $R'_{j_{l-1}} \rightarrow \tau_{\geq l-1} R'_i$ to $R'_{j_l} \rightarrow \tau_{\geq l} R'_i$ amounts to finding j_l large enough such that $R'_{j_l} \rightarrow k \oplus V[l-1]$ is a null homotopic map of simplicial commutative rings. But this is possible, since otherwise it would represent a non-trivial element in $\pi_{-(l-1)} \mathfrak{t}(R', k)$. By induction this group is zero for $l \leq n$ and we obtain the factorization $R_j \rightarrow \tau_{\geq n} R'_i$.

Having proved this main ingredient, we prove the induction step in the statement of (ii) as follows. For surjectivity, we wish to show that any homomorphism $\phi : \pi_n(R'_i) \rightarrow k$ arises from some homomorphism $R'_i \rightarrow k \oplus k[n]$, after possibly increasing i . It is clear that ϕ is induced by a homomorphism $\tau_{\geq n} R'_i \rightarrow k \oplus k[n]$, since $\tau_{\geq n} R'_i \rightarrow \tau_{\leq n} \tau_{\geq n} R'_i \simeq k \oplus V[n]$ with $V = \pi_n(R'_i)$, where this last weak equivalence follows since the objects $k \oplus V[n]$ are “characterized by their homotopy groups” as in the proof of Lemma 2.8.

But then any lift $R'_j \rightarrow \tau_{\geq n} R'_i \rightarrow k \oplus k[n]$ represents ϕ . Injectivity is similar: if $\phi, \psi : R'_i \rightarrow k \oplus k[n]$ induce the same map in π_n , argue that their restrictions to $\tau_{\geq n} R'_i \rightarrow k \oplus k[n]$ are weakly homotopic, i.e. in the same path component of the derived mapping space of simplicial k -algebras (we point out that this is not quite the same as the derived space of morphisms in \mathbf{Art}_k). Then use the factorization.

Statement (iii) is proved by induction on n . For $n = 0$ the “if” part is immediate from the injectivity in (i), and the “only if” part follows from Nakayama’s lemma. For $n > 0$ the induction step follows from (ii). \square

Remark. Just as in the discussion following Lemma 2.8, there is a topological analogue involving p -finite spaces. For any inclusion $X \rightarrow Y$ of such spaces we have a Hurewicz homomorphism

$$H^n(Y, X; k) \rightarrow \mathrm{Hom}_{k[\pi_1(X)]}(\pi_n(Y, X), k).$$

It is true that this homomorphism is an isomorphism in the first degree in which the target is non-zero (even when the spaces are not simply connected). It follows that $H^*(Y, X; k) = 0$ if and only if $(\pi_*(Y, X))^\vee = 0$.

Finally, the following Corollary is an immediate consequence of (4.2) and the Proposition:

Corollary 4.4. *Let R be an object of $\mathrm{pro}\text{-}\mathbf{Art}_k$, and let $\pi_0 R$ be the object of $\mathrm{pro}\text{-}\mathbf{Art}_k$ obtained by applying π_0 level-wise to R . Then one has $\pi_0 \mathfrak{t}(\pi_0 R) = \pi_0 \mathfrak{t}(R)$ and an exact sequence*

$$(4.7) \quad \pi_{-1} \mathfrak{t}(\pi_0 R) \xrightarrow{\iota} \pi_{-1}(\mathfrak{t}R) \rightarrow \mathrm{Hom}_{\pi_0 R}(\pi_1 R, k) \rightarrow \pi_{-2} \mathfrak{t}(\pi_0 R) \rightarrow \pi_{-2}(\mathfrak{t}R) \rightarrow \dots$$

4.2. Cell structures on pro-rings. For simply connected topological space, there always exists a CW approximation with the smallest number of cells consistent with its integral singular homology. The easy direction, which does not use simple connectivity, is that the number of cells is *at least* that much: this follows from a calculation of $H_*(D^n, \partial D^n)$. The other direction is more difficult and requires simple connectivity and the Hurewicz theorem: if $X \rightarrow Y$ is $(n-1)$ -connected, the Hurewicz theorem allows us to lift generators of $H_n(Y, X)$ to maps from $(D^n, \partial D^n)$ along which we may attach more cells to X in order to make the map n -connected.

For morphisms $R \rightarrow R'$ in pro-Art_k the relative tangent complex plays a similar role with respect to the cell attachments described in Definition 3.7. Indeed, if R' is obtained from R by attaching a single n -cell, then it follows from the pullback square that $\pi_{-*}\mathfrak{t}(R', R)$ is one-dimensional in degree $* = n$ and vanishing otherwise. It follows that $\text{Hom}(R, -)$ cannot be obtained from $\text{Hom}(R', -)$ by fewer than $\dim_k(\pi_{-*}\mathfrak{t}(R', R))$ cell attachments. In this section we shall explain why this minimal number of cells can always be realized. In analogy with what happens for spaces, the main ingredient is the Hurewicz type result in Proposition 4.3.

Indeed, if $R \rightarrow R'$ is a morphism in pro-Art_k , and $\pi_{-l}\mathfrak{t}(R, R')$ vanishes for $l < n$ and is finite dimensional for $l = n$, then the Hurewicz theorem proved above implies that the groups $\pi^l\mathfrak{t}(R', R)$ also vanish for $l < n$ lets us lift elements of a dual basis for $\pi_{-n}\mathfrak{t}(R', R)$ to maps $(\Delta^n, \partial\Delta^n) \rightarrow (R', R)$. If R'' is the pro-ring obtained by attaching n -cells to R along these finitely many maps, we obtain a factorization $R \rightarrow R'' \rightarrow R'$, where $\pi_{-*}\mathfrak{t}(R'', R)$ is concentrated in degree n and $\pi_{-n}\mathfrak{t}(R', R) \rightarrow \pi_{-n}\mathfrak{t}(R'', R)$ is an isomorphism. Hence by the long exact sequence, $\pi_{-*}\mathfrak{t}(R', R'')$ vanishes for $* \leq n$ and $\pi_{-*}\mathfrak{t}(R', R'') \rightarrow \pi_{-*}\mathfrak{t}(R', R)$ is an isomorphism for $* > n$. By induction on n , this proves the following result.

Corollary 4.5. *Let $R \rightarrow R'$ be a morphism in pro-Art_k . If $\pi_*\mathfrak{t}(R', R)$ is finite dimensional, then R' is obtained from R by finitely many cell attachments, with one n -cell for each element in a dual basis for $\pi_{-n}\mathfrak{t}(R', R)$. In other words, the functor $\text{Hom}(R, -)$ is naturally weakly equivalent to a functor obtained from the functor $\text{Hom}(R', -)$ by attaching cells (Definition 3.7) precisely $\dim_k(\pi_{-*}\mathfrak{t}(R', R))$ times.*

In particular, if $R \in \text{pro-Art}_k$ is an object with $N = \dim_k(\pi_{-}\mathfrak{t}(R, W)) < \infty$, then $\text{Hom}(R, -)$ is naturally weakly equivalent to a functor obtained from the terminal functor by precisely N cell attachments (Definition 3.7).*

If $\mathcal{F} : \text{Art}_k \rightarrow \text{sSets}$ is pro-representable, then it is sequentially pro-representable if and only if $\pi_\mathfrak{t}(R', R)$ is countable.*

Proof sketch. We already explained how to achieve a cell structure, so it remains to discuss countability. Adjoining a single cell, or a countable number of cells, does not change whether or not the indexing set of the pro-object may be chosen countable, by our constructions. \square

4.3. The tangent complex of a formally cohesive functor. In this section, we will define the *tangent complex* of a formally cohesive functor $\mathcal{F} : \mathbf{Art}_k \rightarrow s\mathbf{Sets}$. It will be a chain complex $\mathfrak{t}\mathcal{F}$, possibly unbounded in both directions. It generalizes the previously defined $\pi_{-n}\mathfrak{t}(R', R)$ for a morphism $R \rightarrow R'$ in \mathbf{Art}_k in the sense that $\pi_{-n}\mathfrak{t}(R', R)$ is the homotopy groups of the mapping cone of $\mathfrak{t}\mathrm{Hom}(R', -) \rightarrow \mathfrak{t}\mathrm{Hom}(R, -)$. Whenever we speak of a “chain complex” we shall always mean one with degree-decreasing boundary map, and vice versa for cochain complexes.

It seems difficult to directly define a chain complex $\mathfrak{t}\mathcal{F}$ from a formally cohesive functor \mathcal{F} . Instead, we construct an essentially equivalent incarnation of it, as a *spectrum* with the structure of a module spectrum over the Eilenberg–MacLane spectrum Hk . We first review the relationship between spectra and chain complexes.

4.3.1. The Dold–Kan correspondence and spectra.

Definition 4.6. A spectrum is a sequence E of based simplicial sets E_n together with maps $\epsilon_n : E_n \rightarrow \Omega E_{n+1}$. It is an Ω -spectrum if all the compositions $E_n \rightarrow \Omega E_{n+1} \rightarrow \Omega \mathrm{Ex}^\infty E_{n+1}$ are weak equivalences. The homotopy groups of a spectrum E are defined for $k \in \mathbb{Z}$ as

$$\pi_k(E) = \mathrm{colim}_{n \rightarrow \infty} \pi_{n+k} E_n.$$

Following Lurie, we can now define the tangent complex of a formally cohesive functor \mathcal{F} as an Ω -spectrum.

Example 4.7. Let \mathcal{F} be a formally cohesive functor, and suppose for convenience of notation that it is Kan valued. Then the tangent complex of \mathcal{F} is the Ω -spectrum $\mathfrak{t}\mathcal{F}$ whose n th space is $\mathcal{F}(k \oplus k[n])$, and whose structure maps are given by the composition

$$\begin{aligned} \mathcal{F}(k \oplus k[n]) &\xrightarrow{\simeq} \mathcal{F}(k) \times_{\mathcal{F}(k \oplus k[n+1])}^h \mathcal{F}(k \oplus \widetilde{k[n]}) \\ &\xrightarrow{\simeq} \{*\} \times_{\mathcal{F}(k \oplus k[n+1])}^h \{*\} = \Omega \mathcal{F}(k \oplus k[n+1]), \end{aligned}$$

where the first equivalence comes from the fact that \mathcal{F} preserves homotopy pullback and the second comes from the fact that $\mathcal{F}(k) \simeq *$.

Next we recall, in Example 4.8 below, how an unbounded chain complex gives rise to an Ω -spectrum via the Dold–Kan correspondence. Recall first that to a simplicial k -module A there is an associated non-negatively graded k -linear chain complex NA : for $p \geq 0$ the p th term is $N_p A = \cap_{i=1}^p \mathrm{Ker}(d_i : A_p \rightarrow A_{p-1})$ and the boundary map $\partial : N_p A \rightarrow N_{p-1} A$ is the restriction of d_0 . If we write $\mathrm{Ch}_+(k)$ for the category of non-negatively graded k -linear chain complexes whose morphisms are the chain maps, then it is a fundamental insight of Dold ([8, §1]) and Kan that the functor

$$N : s\mathrm{Mod}_k \rightarrow \mathrm{Ch}_+(k)$$

is an *equivalence* of categories. The inverse functor $\mathrm{Ch}_+(k) \rightarrow s\mathrm{Mod}_k$ is often called the Dold–Kan functor. The composition

$$\mathrm{Ch}_+(k) \xrightarrow{\text{Dold–Kan}} s\mathrm{Mod}_k \xrightarrow{\text{forget}} s\mathrm{Sets}$$

sends a chain complex (C_*, ∂) to a Kan simplicial set (or topological space, after realization) with basepoint given by 0-element, whose homotopy groups are canonically isomorphic to the homology groups of the chain complex.

If $C = (C_*, \partial) \in \mathrm{Ch}_+(k)$, then the space $|\mathrm{Dold–Kan}(C)|$ is an example of an *infinite loop space*: if ΣC denotes the shifted chain complex with $(\Sigma C)_0 = 0$ and $(\Sigma C)_{n+1} = C_n$, then there is a canonical weak equivalence

$$(4.8) \quad \mathrm{Dold–Kan}(C) \xrightarrow{\simeq} \Omega \mathrm{Dold–Kan}(\Sigma C)$$

reflecting the fact that the homology groups of ΣC are the shifted homology groups of C . By iteration, we get weak equivalences $\mathrm{Dold–Kan}(C) \rightarrow \Omega^n \mathrm{Dold–Kan}(\Sigma^n C)$.

If $C = (C_*, \partial)$ is an object in the category $\mathrm{Ch}(k)$ of *unbounded* chain complexes we can do something similar. Firstly, let $\tau_{\geq 0} C \in \mathrm{Ch}_+(k)$ denote the *truncation*, i.e. the chain complex with $(\tau_{\geq 0} C)_0 = \mathrm{Ker}(\partial : C_0 \rightarrow C_{-1})$ and $(\tau_{\geq 0} C)_n = C_n$ for $n \geq 1$. We may then apply the Dold–Kan functor to the non-negatively graded chain complexes $\tau_{\geq 0}(\Sigma^n C)$ and as in (4.8) above, the underlying based simplicial sets come with weak equivalences

$$\mathrm{Dold–Kan}(\tau_{\geq 0} \Sigma^n C) \xrightarrow{\simeq} \Omega \mathrm{Dold–Kan}(\tau_{\geq 0} \Sigma^{n+1} C).$$

Example 4.8. An unbounded chain complex $C = (C_*, \partial)$ gives rise to an Ω -spectrum with n th space $\mathrm{Dold–Kan}(\tau_{\geq 0} \Sigma^n C)$. The homotopy groups of this spectrum are canonically isomorphic to the homology groups of C .

This defines a functor from $\mathrm{Ch}(k)$ to Ω -spectra which we shall also sometimes call the Dold–Kan functor.

We take the point of view that the Dold–Kan functor from $\mathrm{Ch}(k)$ to spectra defined by the above example is a “forgetful” functor. It remembers enough about a chain complex to recover its homology groups (viz. as the homotopy groups of the spectrum) and in particular it detects quasi-isomorphisms, but it does not remember enough information to recover the k -module structure on these homology groups. In the next few subsections we shall explain how recognize on a given spectrum E the “extra structure” of a weak equivalence $E \simeq \mathrm{Dold–Kan}(C)$ for an unbounded chain complex C ; such an extra structure implies among other things a k -module structure on $\pi_*(E)$. The goal of reviewing this theory is to prove that the tangent complex of a formally cohesive functor \mathcal{F} naturally has this structure: thus, up to weak equivalence, the tangent complex spectrum arises under the Dold–Kan functor from a more fundamental object of $\mathrm{Ch}(k)$ which we consider to be “the” tangent complex of \mathcal{F} .

4.3.2. Γ -sets and Γ -spaces.

Definition 4.9. Let FinSet_* be the category of finite based sets, and let $\Gamma^{\text{op}} \subset \text{FinSet}_*$ be the full subcategory on the objects $\underline{n} = \{*, 1, \dots, n\}$ for $n \geq 0$. A Γ -set is a functor $X : \Gamma^{\text{op}} \rightarrow \text{Sets}$ with $X(\underline{0})$ terminal (i.e., having a single element).

Any Γ -set arises as the restriction of a functor $\text{FinSet}_* \rightarrow \text{Sets}$, and this extension is unique up to unique natural isomorphism. For example, if S is a finite based set we may define

$$X(S) = (X(\underline{n}) \times \text{Iso}_{\text{FinSet}_*}(\underline{n}, S)) / S_n$$

for $|\underline{n}| = |S|$. Henceforth we shall often (tacitly) assume that Γ -spaces $X : \Gamma^{\text{op}} \rightarrow s\text{Sets}$ have been extended to all finite pointed sets in this way.

Definition 4.10 (Segal). A Γ -space is a simplicial Γ -set¹, or in other words a functor $X : \Gamma^{\text{op}} \rightarrow s\text{Sets}$ with $X(\underline{0})$ terminal.

Let $p_i : \underline{n} \rightarrow \underline{1}$ be the morphism with $p_i^{-1}(1) = \{i\}$ and let

$$(4.9) \quad X(\underline{n}) \rightarrow \prod_{i=1}^n X(\underline{1})$$

be the map whose i th coordinate is $X(p_i)$. Then the Γ -space X is special when these maps are weak equivalences for all $n \geq 0$.

Let $\nabla : \underline{2} \rightarrow \underline{1}$ be the unique map with $\nabla^{-1}(1) = \{1, 2\}$. If X is special, then the diagram

$$X(\underline{1}) \times X(\underline{1}) \xleftarrow{(X(p_1), X(p_2))} X(\underline{2}) \xrightarrow{X(\nabla)} X(\underline{1}).$$

induces the structure of an abelian monoid on the set $\pi_0(X(\underline{1}))$, with unit arising from the unique map $\underline{0} \rightarrow \underline{1}$. The Γ -space X is very special if this monoid is a group.

Remark. In Segal's original definition, the requirement that $X(\underline{0})$ be a terminal simplicial set was not included, but many later authors have added this requirement. His less restrictive definition still implies that special Γ -spaces have $X(\underline{0})$ contractible, by the weak equivalence (4.9) for $n = 0$. If X is a Γ -space in this less restrictive sense, the unique morphisms $\underline{0} \rightarrow \underline{n} \rightarrow \underline{0}$ in Γ^{op} gives a canonical factorization of X through a functor into simplicial sets over and under $A = X(\underline{0})$, in which $X(\underline{0})$ is terminal in that category. Now the Γ -space defined by $X'(\underline{n}) = X(\underline{n})/A$ has $X'(A)$ terminal. If A is contractible the natural map $X \rightarrow X'$ is an objectwise weak equivalence, so X' is (very) special if and only if X is. Therefore, since we're mainly interested in special Γ -spaces it does not matter much whether we use the less restrictive notion or not.

It may be helpful to think of the simplicial set $X(\underline{1})$ as the “underlying space” of X , and the fibers of $X(\underline{n}) \rightarrow X(\underline{1})$ as the space of ways to decompose an element as the “sum” of n other elements. A “ Γ -space structure” on a pointed space Y is a Γ -space X and an isomorphism (or perhaps a weak equivalence) $Y \approx X(\underline{1})$.

¹A. Connes has argued that it is better to consider Γ -spaces as simplicial objects in Γ -sets rather than Γ -objects in simplicial sets.

Example 4.11. The Γ -space \mathbb{S} given by the functor which sends \underline{n} to the constant simplicial set \underline{n} is not special. Indeed, $\mathbb{S}(\underline{n})$ has $n + 1$ elements whereas $\prod_1^n \mathbb{S}(\underline{1})$ has 2^n elements.

This particular Γ -space is nevertheless quite important, and is known as the sphere spectrum.

Segal then proves that very special Γ -spaces model Ω -spectra. Let us summarize the construction, following the later exposition by Bousfield and Friedlander.

- (i) If X is any Γ -space, let ΩX be the Γ -space obtained by taking objectwise (simplicial) loop space, i.e. $\underline{n} \mapsto \Omega X(\underline{n})$, where the basepoint comes from the unique map $\underline{0} \rightarrow \underline{n}$.
- (ii) If X is any Γ -space, let BX be the Γ -space defined as

$$(BX)(S) = |[p] \mapsto X(S_p^1 \wedge S)|,$$

where S_p^1 is the pointed simplicial circle, i.e. $S_p^1 = \Delta^1([p]) / (\partial \Delta^1)([p])$, regarded as a functor $\Delta^{\text{op}} \rightarrow \Gamma^{\text{op}}$.

- (iii) The adjoint of the map

$$S^1 \times X(\underline{n}) \cong S^1 \times X(S_1^1 \wedge \underline{n}) \rightarrow (BX)(\underline{n})$$

induces a natural transformation $X \rightarrow \Omega BX$.

More explicitly, at the level of p -simplices, this map sends $(u, v) \in S_p^1 \times X(\underline{n})_p$ to the image of v under the map $\underline{n} \rightarrow S_p^1 \wedge \underline{n}$ induced by $t \mapsto u \wedge t$.

- (iv) Translating into this language, Segal's theorem is firstly that if X is very special (and takes values in Kan complexes), then $X \rightarrow \Omega BX$ induces a weak equivalence $X(\underline{n}) \rightarrow \Omega(\text{Ex}^\infty(BX(\underline{n})))$, secondly that if X is special then so is BX , and thirdly that $(BX)(\underline{1})$ is n -connected when $X(\underline{1})$ is $(n - 1)$ -connected.

In particular when X is very special the space $X(\underline{1})$ comes with canonical deloopings $(BX)(\underline{1})$, $(B^2X)(\underline{1})$, \dots , which depend only on the Γ -space structure, and hence $X(\underline{1})$ is canonically the 0-space in an Ω -spectrum. If X is special but not very special, the “group completion” of $X(\underline{1})$ (i.e. $\Omega B(X(\underline{1}))$) is the zero-space of an Ω -spectrum.

Definition 4.12 (Bousfield–Friedlander). *Let X be any Γ -space, and define*

$$\pi_k(X) = \text{colim}_n \pi_{n+k} B^n X.$$

A map $X \rightarrow Y$ of Γ -spaces is a stable equivalence if it induces an isomorphism in all homotopy groups.

4.3.3. Smash product, Γ -rings and module spectra. Segal's category of Γ -spaces was later studied further by Bousfield and Friedlander, who constructed a closed model category structure on Γ -spaces whose weak equivalences are the stable equivalences from Definition 4.12 and proved that its homotopy category is equivalent to the homotopy category of connective spectra in other known models of that category. (See also [27, §4, Chapter II]). Later, Lydakis [23] gave a model for the smash product $E \wedge F$ of two spectra E and F arising from Γ -spaces: for Γ -spaces

defined on all finite based sets, the smash product has the universal property that a map of Γ -spaces $E \wedge F \rightarrow G$ is the same (up to isomorphism) as maps

$$E(S) \times F(T) \rightarrow G(S \wedge T)$$

forming a natural transformation of functors of $(S, T) \in \Gamma^{\text{op}} \times \Gamma^{\text{op}}$, where \wedge is the smash product of based sets.

Example 4.13. *For any Γ -space X , there are canonical maps*

$$\underline{\mathbb{N}} \times X(T) \rightarrow X(\underline{\mathbb{N}} \wedge T),$$

whose restriction to $\{i\} \times X(T)$ is given by the map $X(T) \rightarrow X(\underline{\mathbb{N}} \wedge T)$ induced by the injection $T \cong \{, i\} \wedge T \rightarrow \underline{\mathbb{N}} \wedge T$. This produces a canonical map of Γ -spaces*

$$\mathbb{S} \wedge X \rightarrow X,$$

which is in fact an isomorphism.

Definition 4.14. (i) A Γ -ring is a triple (R, μ, ν) consisting of a Γ -space R , a (strictly!) associative and commutative map $\mu : R \wedge R \rightarrow R$, and a unit map $\nu : \mathbb{S} \rightarrow R$ satisfying that $\mu \circ (\nu \wedge \text{id}) : \mathbb{S} \wedge R \rightarrow R$ agrees with the canonical isomorphism.

(ii) A module over a Γ -ring R consists of a Γ -space M together with a map $R \wedge M \rightarrow M$ satisfying the obvious axioms.

For example, any Γ -space is canonically an \mathbb{S} -module. A Γ -ring give rise a connective *ring spectrum*, although the two notions are not quite the same [21]. Important examples come from the *Eilenberg–MacLane spectrum* construction, which associates a spectrum HV to a simplicial abelian group V .

Definition 4.15. (i) Let V be an abelian group. The Eilenberg–MacLane space HV is the Γ -set defined by

$$HV(S) = H_0(S, *; V) \cong \prod_{S \setminus \{*\}} V.$$

*The description as relative singular homology $H_0(S, *; V)$ makes the functoriality clear, and it is clear from the product description that it is special (the map (4.9) is a bijection of sets). Then $\pi_0(HV(\underline{1})) = V$ and the monoid structure agrees with vector space addition so HV is very special.*

(ii) If V is a simplicial abelian group, define $HV : \Gamma^{\text{op}} \rightarrow \text{sSets}$ by applying the previous construction degreewise.

Example 4.16. *For abelian groups V, W the map*

$$HV(S) \times HW(T) = H_0(S, *; V) \times H_0(T, *; W) \rightarrow H_0(S, *; V) \otimes H_0(T, *; W) \\ \xrightarrow{\cong} H_0(S \wedge T, *; V \otimes W),$$

obtained by composing the “Künneth isomorphism” and the canonical bilinear map defining the tensor product, is a natural transformation of functors of the based finite sets S, T , and hence defines a map of spectra

$$(4.10) \quad HV \wedge HW \rightarrow H(V \otimes W).$$

This construction works degreewise for simplicial abelian groups, and the map is natural in the simplicial abelian groups V and W .

If k is a simplicial commutative ring, then the multiplication $k \otimes k \rightarrow k$ gives rise to a map $H(k \otimes k) \rightarrow Hk$, which composed with (4.10) makes Hk into a Γ -ring. If V is a simplicial k -module, then HV inherits an Hk -module structure in a similar way.

A map $X \rightarrow Y$ of Hk -module spectra is a map of Γ spaces which commutes (strictly) with the module structure maps. Such a map is a weak equivalence if the underlying map of Γ -spaces is (i.e. is a stable equivalence in the sense of Definition 4.12; we emphasize that this is *not* the same as objectwise weak equivalence of functors $\Gamma^{\text{op}} \rightarrow s\text{Sets}$). It is a result of Stefan Schwede that this notion can be extended to a full-fledged model category structure. The following result plays an important technical role in this paper.

Theorem 4.17 (Robinson, Schwede). *Let k be a (possibly simplicial) commutative ring. The Eilenberg–MacLane functor induces an equivalence of categories from the homotopy category of simplicial k -modules to the homotopy category of Hk modules. In fact, this functor is part of a Quillen equivalence between model categories.*

In particular (for $k = \mathbb{Z}$) it induces an equivalence from simplicial abelian groups to $H\mathbb{Z}$ -modules.

By this result, the homotopy theory of simplicial k -modules (or equivalently non-negatively graded chain complexes of k -modules) is “equivalent” to the homotopy theory of Hk -modules. In particular there is for each Hk -module E a “corresponding” simplicial k -module V and a zig-zag of weak equivalences of Hk -modules between HV and E . The usefulness of this result is that an object may arise quite naturally and explicitly as an Hk -module E but not explicitly as a simplicial k -module. For example, in this paper this is the case for the tangent complex of a formally coherent functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$.

Although we shall not strictly need it, let us briefly discuss the extent to which this relation between Hk -modules and simplicial k -modules may be made functorial. In fact, Schwede promotes the relation to a Quillen equivalence of model categories, where $V \mapsto HV$ is the right adjoint. He defines a functor in the other direction, sending an Hk -module E to a simplicial k -module LE . The functor L is left adjoint to H in the strict sense, and has an explicit and rather simple construction which we shall omit here. The *homotopy* inverse to H then sends an Hk -module spectrum E to the simplicial k -module $Lc(E)$, where c is a cofibrant replacement functor on Hk -module spectra (in a certain model category structure, see Schwede’s paper for details). Such a functor c is proven to exist using the small object argument, but unfortunately there does not seem to be a known explicit formula. Consequently we do not know an explicit functorial formula for the simplicial k -module “corresponding” to an Hk -module spectrum E , but nevertheless it may be useful (or at least consoling) to remember the direction of the arrows

in the comparison zig-zags: to an Hk -module E the “corresponding” simplicial k -module is $V = Lc(E)$, and we have natural weak equivalences of Hk -modules

$$(4.11) \quad E \xleftarrow{\simeq} cE \xrightarrow{\simeq} HV.$$

4.3.4. Non-connective spectra. A spectrum $E = (E_n, \epsilon_n)_{n \in \mathbb{N}}$ is *connective* if $\pi_k(E) = 0$ for $k < 0$. For Ω -spectra this is equivalent to each space E_n being $(n - 1)$ -connected. As already alluded to, connective Ω -spectra are, up to weak equivalence, the same as very special Γ -spaces. More precisely there are functors in both directions given as follows.

Example 4.18. Any Γ -space X gives rise to a connective spectrum with $E_n = (B^n X)(\underline{1})$. It is an Ω -spectrum when X is very special (after possibly applying fibrant replacement to the E_n s).

Conversely, to a spectrum E there is an associated very special Γ -space defined by

$$S \mapsto \operatorname{colim}_{n \rightarrow \infty} \Omega^n \operatorname{Ex}^\infty(S \wedge E_n).$$

Here $S \wedge E_n = \vee^{S \setminus \{*\}} E_n$ is the smash product of pointed simplicial sets, where S is regarded as a constant simplicial set.

These constructions are inverse up to homotopy, and give an equivalence of categories between the homotopy category of very special Γ -spaces and connective Ω -spectra.

To get a model for *all* spectra, including those with homotopy groups in negative degrees, we may consider spectrum objects in Γ -spaces.

Definition 4.19. A Γ -spectrum is a sequence E consisting of Γ -spaces E_n and maps $\epsilon_n : E_n \rightarrow \Omega E_{n+1}$ of Γ -spaces.

This notion of Γ -spectrum is somewhat uncommon, probably for the following reason. The forgetful functor from Γ -spectra to spectra induced by sending E to the spectrum with n th space $E_n(\underline{1})$ induces an equivalence of homotopy categories, so in this sense the “ Γ -structure” is redundant. Nevertheless, it seems convenient to keep track of this extra redundant data, for instance in dealing with Hk -module structures.

Definition 4.20. A (non-connective) Hk -module spectrum is a Γ -spectrum E together with maps of Γ -spaces $\mu_n : Hk \wedge E_n \rightarrow E_n$ making E_n into an Hk module, such that the maps $\epsilon_n : E_n \rightarrow \Omega E_{n+1}$ are maps of Hk modules.

As already explained, the Dold–Kan functor may be promoted to a functor from unbounded chain complexes to spectra. In fact this functor naturally takes values in Hk -module spectra.

Example 4.21. Recall from Example 4.16 that the Eilenberg–MacLane functor takes a simplicial k -module V to an Hk -module HV . If $C \in \operatorname{Ch}_+(k)$ is a non-negatively graded k -linear chain complex, then we obtain an Hk -module

$H(\text{Dold-Kan}(C))$. If $C \in \text{Ch}(k)$ is an unbounded chain complex, then we obtain an Hk -module spectrum with n th space

$$H(\text{Dold-Kan}(\tau_{\geq 0}\Sigma^n C)).$$

It remains to define the structure maps in this Hk -module spectrum. First recall that for simplicial k -modules A and B the Alexander–Whitney formula gives rise to a map of k -linear chain complexes $N(A \otimes_k B) \rightarrow NA \otimes_k NB$. If we write kS^1 for the free k -module on the simplicial circle, modulo the span of the basepoint and its degeneracies, and apply the Alexander–Whitney map in the special case $A = kS^1$ and $B = \text{Dold-Kan}(D)$ for some non-negatively graded k -linear chain complex D , we get a natural transformation

$$N(kS^1 \otimes \text{Dold-Kan}(D)) \rightarrow N(kS^1) \otimes D \cong \Sigma D.$$

Applying the Dold–Kan functor to this map, and using that it is an inverse functor to N up to isomorphism, we then get a natural transformation of simplicial k -modules

$$kS^1 \otimes \text{Dold-Kan}(D) \rightarrow \text{Dold-Kan}(\Sigma D).$$

The loop-space functor Ω , when regarded as a endo-functor of simplicial k -modules, is right adjoint to the functor $kS^1 \otimes -$, so the above map is adjoint to a map

$$(4.12) \quad \text{Dold-Kan}(D) \rightarrow \Omega(\text{Dold-Kan}(\Sigma D))$$

of simplicial k -modules, natural in the non-negatively graded k -linear chain complex D . Finally, substitute $D = \tau_{\geq 0}\Sigma^n C$ into (4.12) and use that the inclusion

$$\text{Dold-Kan}(\Sigma\tau_{\geq 0}\Sigma^n C) = \text{Dold-Kan}(\tau_{\geq 1}\Sigma^{n+1}C) \subset \text{Dold-Kan}(\tau_{\geq 0}\Sigma^{n+1}C)$$

becomes an isomorphism after applying Ω . This defines a natural transformation between simplicial k -modules, and the Eilenberg–MacLane functor turns this into a map of Hk -modules.

Theorem 4.22. *The functor from unbounded k -linear chain complexes to Hk -module spectra defined in the above example induces an equivalence of homotopy categories.*

Proof sketch. This is just assembling other equivalences: the Dold–Kan functor gives an equivalence between $\text{Ch}_+(k)$ and $s\text{Mod}_k$, Robinson and Schwede’s results give an equivalence between $s\text{Mod}_k$ and Hk -modules. There is an induced equivalence between “ Ω -spectrum objects” in $\text{Ch}_+(k)$ and Hk module spectra. Spelling this out, we have an equivalence from $\text{Ch}(k)$ to Hk -module spectra. \square

The above functor from $\text{Ch}(k)$ to Hk -module spectra, defined explicitly in terms of the Dold–Kan functor and the Eilenberg–MacLane functor, again has a homotopy inverse *functor* which turns an Hk -module spectrum E into an unbounded chain complex. This inverse functor is defined by turning each Hk -module E_n into a k -linear positively graded chain complex $Lc(E_n)$; the maps $E_n \rightarrow \Omega E_{n+1}$ then induces maps of chain complexes $\Sigma Lc(E_n) \rightarrow Lc(E_{n+1})$ and we may form a chain complex as the colimit (or homotopy colimit) of $\Sigma^{-n}Lc(E_n)$. Up to weak

equivalence we get the Hk -module spectrum back by applying the functor in Example 4.21; conversely, turning an unbounded k -linear chain complex into an Hk -module spectrum and then back to a chain complex results in a chain complex quasi-isomorphic to the one we started with. Unfortunately the functor from Hk -module spectra to chain complexes is not nearly as explicit as the one in the other direction, since it involves the inexplicit cofibrant approximation $c(E_n) \rightarrow E_n$ of Hk -modules.

4.4. The tangent complex of a formally cohesive functor as an Hk module spectrum. Let us now finally return to formally cohesive functors. If $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ is such a functor, we have already explained how to define $\mathfrak{t}\mathcal{F}$ as an Ω -spectrum, whose n th space is

$$(\mathfrak{t}\mathcal{F})_n = \mathcal{F}(k \oplus k[n]).$$

This is an Ω -spectrum by the cohesiveness assumption, and is usually not connective. The goal of this section (and the goal of recalling all the Γ -space machinery!) is to explain why the based simplicial sets $(\mathfrak{t}\mathcal{F})_n$ have canonical Γ -space structures, and why the resulting Γ -spectrum $\mathfrak{t}\mathcal{F}$ is naturally an Hk -module spectrum.

Lemma 4.23. *Let $\mathcal{F} : \mathbf{Art}_k$ be formally cohesive and let V be a simplicial k -module with $\pi_*(V)$ finite dimensional. Then the Γ -space $S \mapsto \mathcal{F}(k \oplus HV(S))$ is special.*

Proof. Strictly speaking $\mathcal{F}(k) = \mathcal{F}(k \oplus HV(\underline{0}))$ need not be terminal, so we should first quotient it out, as described in the remark following Definition 4.10.

If W_1, \dots, W_n are simplicial k -modules, the projection maps $p_j : \bigoplus_i W_i \rightarrow W_j$ induce a map

$$\mathcal{F}(k \oplus \bigoplus_{i=1}^n W_i) \rightarrow \prod_{i=1}^n \mathcal{F}(k \oplus W_i),$$

and by induction the formal cohesiveness of \mathcal{F} implies that these are weak equivalences. Applying this to $W_1 = \dots = W_n = V$ we see that the map

$$\mathcal{F}(k \oplus HV(\underline{n})) \rightarrow \prod_{i=1}^n \mathcal{F}(k \oplus HV(\underline{1}))$$

is a weak equivalence and hence $S \mapsto \mathcal{F}(k \oplus HV(S))$ is special. \square

The following Lemma and its proof shows the advantage of working with Hk -modules as opposed to simplicial k -modules.

Lemma 4.24. *Let S and T be based finite sets and let $\mu : Hk(S) \times HV(T) \rightarrow HV(S \wedge T)$ be the maps defining the multiplication $Hk \wedge HV \rightarrow HV$ in the Hk -module structure on HV . From the definition of μ we see that for each $v \in Hk(S) = \prod^{S \setminus \{*\}} k$ the map $\mu(v, -) : HV(T) \rightarrow HV(S \wedge T)$ is a map of simplicial k -modules, and hence induces maps*

$$\mathcal{F}(k \oplus -)(\mu(v, -)) : \mathcal{F}(k \oplus HV(T)) \rightarrow \mathcal{F}(k \oplus HV(S \wedge T)),$$

and as $v \in Hk(S)$ varies, these assemble to a map

$$Hk(S) \times \mathcal{F}(k \oplus HV(T)) \rightarrow \mathcal{F}(k \oplus HV(S \wedge T))$$

making the special Γ -space $T \mapsto \mathcal{F}(k \oplus HV(T))$ into an Hk -module spectrum.

Proof. This follows from the functoriality of the constructions involved. \square

We have not yet verified that the Γ -spaces constructed above are very special. We shall do that by providing deloopings, which are in fact (usually) non-connective. We shall only need this in the case $V = k[n]$, but the following Lemma may be generalized to other V .

Lemma 4.25. *Let $(\mathfrak{t}\mathcal{F})_n$ be the special Γ -space $S \mapsto \mathcal{F}(k \oplus H(k[n])(S))$, for $n \geq 0$. The natural maps*

$$\mathcal{F}(k \oplus H(k[n])(S)) \rightarrow \Omega\mathcal{F}(k \oplus H(k[n+1])(S)),$$

arising from the weak equivalence from $k \oplus H(k[n])(S)$ to the homotopy pullback of $k \rightarrow k \oplus H(k[n+1])(S) \leftarrow k$, define maps of special Γ -spaces $(\mathfrak{t}\mathcal{F})_n \rightarrow \Omega(\mathfrak{t}\mathcal{F})_{n+1}$ which are Hk -module maps and also weak equivalences.

Proof. The maps are weak equivalences for each S by the formal cohesiveness of \mathcal{F} . It is easy to check that the deloopings commute (strictly, as usual) with the Hk -module structure maps. \square

Definition 4.26. *The tangent complex $\mathfrak{t}\mathcal{F}$ is the chain complex associated (as in Example 4.21) to the Hk -module spectrum defined by the Γ -space $S \mapsto \mathcal{F}(k \oplus Hk(S))$ and the deloopings $\mathcal{F}(k \oplus H(k[n])(S))$ provided above.*

In many cases of interest the space $\mathcal{F}(k \oplus k)$ is discrete, and then the chain complex $\mathfrak{t}\mathcal{F}$ has homology groups only in non-positive degrees. For example, this is the case for the functors $\mathcal{F}_{R,\bar{\rho}}$ from Example 3.10.

The following important example explains the connection to the usual *cotangent complex* of Quillen and Illusie: To a map $A \rightarrow B$ of commutative rings, or indeed of simplicial commutative rings, and a B -module M , we can define André–Quillen cohomology groups $D_A^i(B, M)$; if $A \rightarrow B$ is cofibrant, this is the cohomology of the cosimplicial abelian group obtained by computing (levelwise) A -linear derivations of B with targets in M . Similarly one dually defines a simplicial B -module $L_{B/A}$, the “cotangent complex.” Thus if A, B are usual rings and M a usual B -module, $D_A^0(B, M)$ is the set of derivations $B \rightarrow M$ and $D_A^1(B, M)$ classifies commutative A -algebra extensions $M \rightarrow? \rightarrow B$, whereas $\pi_0 L_{B/A} \simeq \Omega_{B/A}$. For details see [27].

Example 4.27. *If R is a cofibrant simplicial commutative ring and $\bar{\rho} : \pi_0 R \rightarrow k$ is a homomorphism, we may consider the functor $\mathcal{F}_{R,\bar{\rho}} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ from Example 3.10. By (4.1) and the subsequent discussion,*

$$\pi_{-i} \mathfrak{t}\mathcal{F}_{R,\bar{\rho}} \simeq D_{\mathbb{Z}}^i(R, k).$$

Indeed, the tangent complex of $\mathcal{F}_{R,\overline{p}}$ is quasi-isomorphic to

$$(4.13) \quad \mathrm{Hom}_{\mathrm{Ch}(k)}(\mathrm{Dold}\text{--}\mathrm{Kan}(L_{R/\mathbb{Z}} \otimes_R k), k),$$

where the Hom is the internal hom of chain complexes.

4.5. Constructions on cohesive functors and their effect on tangent complexes.

In this section we discuss various constructions which produce new cohesive functors out of old ones, and discuss the effect of these functors on tangent complexes. Both of the following two Lemmas are special cases of the more general statement that the class of formally cohesive functors are closed under taking objectwise homotopy limits.

Lemma 4.28. *Let $\mathcal{F} : \mathrm{Art}_k \rightarrow s\mathrm{Sets}$ be formally cohesive and let X be a simplicial set, or even a pro-simplicial set. Then the functor $A \mapsto \mathrm{Hom}(X, \mathcal{F}(A))$ is formally cohesive, where Hom denotes the space of unbased maps. If \mathcal{F} takes values in pointed spaces and X is pointed, then the same is true for the space of based maps.*

As usual, if $X = (\alpha \mapsto X_\alpha)$ is a pro-simplicial set, we define $\mathrm{Hom}(X, -)$ to be the colimit $\mathrm{colim}_\alpha \mathrm{Hom}(X_\alpha, -)$.

Proof. $\mathrm{Hom}(X, -)$ preserves homotopy pullbacks. \square

Lemma 4.29. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_{01} : \mathrm{Art}_k \rightarrow s\mathrm{Sets}$ be homotopy invariant functors, equipped with natural transformations $\mathcal{F}_0 \rightarrow \mathcal{F}_{01} \leftarrow \mathcal{F}_1$, and define \mathcal{F} by the objectwise homotopy pullback*

$$\mathcal{F}(A) = \mathcal{F}_0(A) \times_{\mathcal{F}_{01}(A)}^h \mathcal{F}_1(A).$$

If $\mathcal{F}_0, \mathcal{F}_1$, and \mathcal{F}_{01} are formally cohesive, then so is \mathcal{F} . More generally, if $\mathcal{F}_0, \mathcal{F}_1$, and \mathcal{F}_{01} preserve homotopy pullback, then so does \mathcal{F} .

Proof. This is a formal consequence of homotopy limits commuting with each other up to weak equivalence; in particular any homotopy limit preserves homotopy pullback. \square

Lemma 4.30. *Let $\mathcal{F} : \mathrm{Art}_k \rightarrow s\mathrm{Sets}$ be formally cohesive.*

- (i) *The tangent complex of the “forgetful” functor $A \mapsto \mathfrak{m} = \mathrm{Ker}(A \rightarrow k)$ is one-dimensional and concentrated in degree 0 (i.e. the corresponding Hk -module spectrum is equivalent to Hk as an Hk -module spectrum).*
- (ii) *If \mathcal{F} takes values in pointed simplicial sets, then $\mathfrak{t}\Omega\mathcal{F} \simeq \Sigma^{-1}\mathfrak{t}\mathcal{F}$.*
- (iii) *For a space X , we have $\mathfrak{t}\mathrm{Hom}(X, \mathcal{F}(-)) \simeq C^*(X; \mathfrak{t}\mathcal{F})$. Similarly when X is a pro-space, where both sides are then interpreted as the direct limits of maps from finite levels of the pro-object.*
- (iv) *The tangent complex takes homotopy pullback squares of functors are taken to pullback squares of spectra. Hence for $\mathcal{F} = \mathcal{F}_0 \times_{\mathcal{F}_{01}}^h \mathcal{F}_1$ as in Lemma 4.29 above, there is a (co)fiber sequence of spectra*

$$\mathfrak{t}\mathcal{F} \rightarrow \mathfrak{t}\mathcal{F}_0 \times \mathfrak{t}\mathcal{F}_1 \rightarrow \mathfrak{t}\mathcal{F}_{01}$$

and a corresponding “Mayer–Vietoris” sequence in homotopy groups

$$\cdots \rightarrow \pi_{-n} \mathfrak{t}\mathcal{F} \xrightarrow{((f_0)_*, (f_1)_*)} \pi_{-n} \mathfrak{t}\mathcal{F}_0 \oplus \pi_{-n} \mathfrak{t}\mathcal{F}_1 \xrightarrow{(p_0)_* - (p_1)_*} \pi_{-n} \mathfrak{t}\mathcal{F}_{01} \rightarrow \pi_{-(n+1)} \mathfrak{t}\mathcal{F} \rightarrow \cdots,$$

where $p_i : \mathcal{F}_i \rightarrow \mathcal{F}_{01}$ are the specified natural transformations and $f_i : \mathcal{F} \rightarrow \mathcal{F}_i$ is the induced projections from the homotopy fiber product to its factors.

We will often use (iv) in the following form: a homotopy commutative square of functors

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{F}_0 \\ \downarrow & & \downarrow \\ \mathcal{F}_1 & \longrightarrow & \mathcal{F}_{01} \end{array}$$

is an object-wise homotopy pullback square exactly when the induced map

$$\mathfrak{t}\mathcal{G} \rightarrow \mathrm{hofib}(\mathfrak{t}\mathcal{F}_0 \oplus \mathfrak{t}\mathcal{F}_1 \longrightarrow \mathfrak{t}\mathcal{F}_{01})$$

is an isomorphism on homotopy groups. Indeed by our discussion above, the latter condition is satisfied exactly when $\mathcal{G} \rightarrow \mathcal{F}_0 \times_{\mathcal{F}_{01}}^h \mathcal{F}_1$ induces an isomorphism on homotopy groups of tangent complexes, thus is an equivalence by Lemma 3.11.

Proof. For (i), clearly the functor takes $k \oplus k[n]$ to $k[n]$ and hence $\pi_i \mathfrak{t}$ has the same homotopy groups as Hk so they must be weakly equivalent.

The remaining statements are special cases of the general statement that if C is any category and $\mathcal{F} = \mathrm{holim}_{c \in C} \mathcal{F}_c$, then $\mathfrak{t}\mathcal{F} = \mathrm{holim}_{c \in C} \mathfrak{t}\mathcal{F}_c$. This follows because (homotopy) limits of Ω -spectra may be computed levelwise. \square

Remark. By combining these results, we see that the tangent complex of the functor $A \mapsto s\mathrm{Sets}_*(X, \mathfrak{m})$ is $C^*(X, *; k)$.

These result shows that the tangent complex behaves well with respect to homotopy pullback of functors, or equivalently derived tensor product of (pro-)simplicial rings. By contrast, the behaviour under pullback of rings is much more complicated. Similarly, the tangent complexes of $k \oplus k[n]$ are quite complicated.

Later we shall define for any algebraic group G a functor $A \mapsto BG(A)$. The following results will let us prove that that functor preserves homotopy pullbacks.

Lemma 4.31. *For a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ C & \xrightarrow{j} & D \end{array}$$

with all four spaces are path connected, and such that either $\pi_1(B) \rightarrow \pi_1(D)$ or $\pi_1(C) \rightarrow \pi_1(D)$ is surjective, the square is homotopy cartesian if and only if the induced square of loop spaces is homotopy cartesian.

Proof. To see that the induced map $A \rightarrow B \times_D^h C$ is a weak equivalence, we check that $\pi_i(A) \rightarrow \pi_i(B \times_D^h C)$ is a bijection for all i and for all basepoints. In fact, we claim that for $i = 0$ both homotopy sets have exactly one element. Given that, the bijection on higher homotopy groups follow since $\pi_i A = \pi_{i-1} \Omega A$ and $\pi_i(B \times_D^h C) = \pi_{i-1}(\Omega B \times_{\Omega D}^h \Omega A)$.

$\pi_0(A)$ has one element by assumption. The set $\pi_0(B \times_D C)$ fits in a short exact sequence of pointed sets $\pi_0(\text{hofib}(j)) \rightarrow \pi_0(B \times_D C) \rightarrow \pi_0 B$, where $\pi_0 B$ has one element by assumption, and $\text{hofib}(j)$ has one element since $\pi_1 C \rightarrow \pi_1(D)$ is surjective and D is path connected. \square

Corollary 4.32. *Let $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}_*$ be a homotopy invariant functor into based simplicial sets, such that $\mathcal{F}(A)$ is path connected for all A , that $A \mapsto \Omega \mathcal{F}(A)$ preserves homotopy pullback, and such that $\pi_0 \Omega \mathcal{F}(A) \rightarrow \pi_0 \Omega \mathcal{F}(B)$ is surjective whenever $\pi_0 A \rightarrow \pi_0 B$ is surjective. Then \mathcal{F} preserves homotopy pullback.*

4.6. Lurie’s derived Schlessinger criterion. The following result is from [22, 6.2.14]. In our applications to number theory, we will only use the “countable-dimensional” case of the result. In the following we shall use the notation $\pi_i \mathfrak{t}\mathcal{F}$ for the i th homology group of the chain complex $\mathfrak{t}\mathcal{F}$ (which is the i th homotopy group of the corresponding Hk -module spectrum).

Theorem 4.33. *Let $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ be formally cohesive. Then \mathcal{F} is pro-representable if and only if the k -vector space $\pi_i \mathfrak{t}\mathcal{F}$ vanishes for $i > 0$.*

If the k -vector spaces $\pi_i \mathfrak{t}\mathcal{F}$ have countable dimension, then the pro-representing object may be chosen to have countable indexing category.

Recall that a spectrum or chain complex E is called *connective* if $\pi_i E = 0$ for $i < 0$ and is called *co-connective* if $\pi_i E = 0$ for $i > 0$. Thus the theorem asserts that a functor is pro-representable if and only if the tangent complex is co-connective; this happens precisely when $\mathcal{F}(k \oplus k[0])$ is homotopy discrete. This is a variant of Theorem 6.2.13 in Lurie’s thesis and is proved by essentially the same argument. Since his setup and assumption are mildly different from ours (e.g. we do not discuss any “Noetherian-ness” of the representing pro-system, and correspondingly we do not assume the homotopy groups of the tangent complex are finite dimensional), we shall outline the proof.

Sketch of proof of Theorem 4.33. In this proof we shall write $\mathfrak{t}\mathcal{F}$ for the Hk -module spectrum version of the tangent complex. It follows from Example 4.27 that the tangent complex of any representable functor is co-connective. The tangent complex takes filtered homotopy colimits of functors to filtered homotopy colimits of spectra, so the tangent complex of any pro-representable functor is co-connective.

Conversely, suppose $\pi_i \mathfrak{t}\mathcal{F} = 0$ for $i > 0$, i.e. that $\pi_i \mathcal{F}(k \oplus k[n]) = 0$ for $i > n$. We shall produce a natural weak equivalence from a filtered homotopy colimit of representable functors to \mathcal{F} . Without loss of generality, we may assume that \mathcal{F} is a simplicially enriched functor and takes Kan values.

The construction is by a (generally transfinite) recursive recipe, providing an “improvement” to any pair (R, ι) consisting of a cofibrant $R \in \text{Art}_k$ and a zero-simplex $\iota \in \mathcal{F}(R)$. Suppose given an integer $n \geq 0$, a finite dimensional k -vector space V and a point in the homotopy fiber of the map $\text{Hom}(R, k \oplus V^\vee[n+1]) \rightarrow \mathcal{F}(k \oplus V^\vee[n+1])$, given by a diagram

$$(4.14) \quad \begin{array}{ccc} \partial\Delta[1] & \xrightarrow{f} & \text{Hom}(R, k \oplus V^\vee[n+1]) \\ \downarrow & & \downarrow \iota \\ \Delta[1] & \xrightarrow{h} & \mathcal{F}(k \oplus V^\vee[n+1]), \end{array}$$

where f and h are pointed maps. We then construct an object $R' \in \text{Art}_k$ and a morphism $R' \rightarrow R$ by the cartesian square

$$\begin{array}{ccc} R' & \longrightarrow & k \oplus \widetilde{V^\vee[n+1]} \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & k \oplus V^\vee[n+1], \end{array}$$

where as usual we write $k \oplus \widetilde{V^\vee[n+1]} = k \oplus (V^\vee \otimes_k \widetilde{k[n+1]}) \simeq k$. Since f induces a surjection on π_0 and \mathcal{F} is formally cohesive we obtain a fiber sequence $\mathcal{F}(R') \rightarrow \mathcal{F}(R) \rightarrow \mathcal{F}(k \oplus V^\vee[n+1])$, where the second map sends ι to $f_*(\iota)$. The path $h : D^1 \rightarrow \mathcal{F}(k \oplus V^\vee[n+1])$ then provides a lift of ι to the homotopy fiber of $\mathcal{F}(R) \rightarrow \mathcal{F}(k \oplus V^\vee[n+1])$, and hence a homotopy lift of ι to $\iota' \in \mathcal{F}(R')$. As we shall explain next, this pair (R', ι') can be regarded as an “improvement” of (R, ι) .

In terms of represented functors, we have constructed a natural diagram of functors $\text{Art}_k \rightarrow s\text{Sets}$

$$\begin{array}{ccc} \text{Hom}(R, -) & \xrightarrow{\iota} & \mathcal{F} \\ \downarrow & \nearrow & \parallel \\ \text{Hom}(R', -) & \xrightarrow{\iota'} & \mathcal{F} \end{array}$$

where the double arrow denotes a natural homotopy. Evaluating at $k \oplus k[n]$ these functors define the spectrum underlying the cotangent complex, and we have an induced homotopy commutative diagram

$$\begin{array}{ccccc} \mathfrak{t}R & \xrightarrow{\iota} & \mathfrak{t}\mathcal{F} & \longrightarrow & C \\ \downarrow & & \parallel & & \downarrow \\ \mathfrak{t}R' & \xrightarrow{\iota'} & \mathfrak{t}\mathcal{F}(-) & \longrightarrow & C' \end{array}$$

where C and C' are the cofibers in the category of Hk -module spectra (they should be thought of as the *relative* cotangent complexes of \mathcal{F} relative to R or R'). The homotopy type of the maps (f, h) from (4.14) correspond bijectively to k -linear

maps $V \rightarrow \pi_{-n}C$, and by construction of (R', ι') the composition $V \rightarrow \pi_{-n}C \rightarrow \pi_{-n}C'$ is zero.

By iterating this procedure, possibly transfinitely many times, we obtain an inverse system of simplicial rings R_j , and a point in the homotopy inverse limit of $\mathcal{F}(R_j)$, such that the corresponding relative cotangent complexes C_j have contractible direct limit (any element at some stage is always killed at some later stage). Hence we have produced a natural weak equivalence $\text{hocolim}_{j \in J} \text{Hom}(R_j, -) \rightarrow \mathcal{F}$, as desired. \square

4.7. Non-reduced functors and local systems. In this section we discuss how to generalize the definitions and results above to functors $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ which preserve homotopy pullback, but which do not necessarily have $\mathcal{F}(k)$ contractible and hence is not formally cohesive.

Such functors are important for our later constructions: for example, in §5 we shall define, for G an algebraic group over $W(k)$, a functor $A \mapsto BG(A)$ which is homotopy invariant and preserves homotopy pullbacks, but it is not reduced. Our representation functor is a modification of $A \mapsto s\text{Sets}(X, BG(A))$, and it is convenient first study $BG(A)$ and then afterwards study the effect of taking mapping space from X .

Rather than forcing a functor $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}$ preserving homotopy pullbacks to have contractible value at k , we change the target category in order for the object $\mathcal{F}(k)$ to be *homotopy terminal* (i.e. have contractible derived mapping space from any other object). Since any object $A = (A \rightarrow k) \in \text{Art}_k$ comes with a unique morphism to $k = (\text{id} : k \rightarrow k)$ there is an induced map $\mathcal{F}(A) \rightarrow \mathcal{F}(k)$, and we may regard \mathcal{F} as taking values in the over category $s\text{Sets}_{/Z}$ for $Z = \mathcal{F}(k)$, or when technically convenient $Z = \text{Ex}^\infty(\mathcal{F}(k))$. In this setting it still makes sense to define the tangent complex of \mathcal{F} , but instead of a k -linear chain complex (or equivalently an Hk -module spectrum), it will be a *local system* of k -linear chain complexes on Z , in the following sense.

Definition 4.34. *The category $\text{Simp}(Z)$ of simplices of a simplicial set Z has objects pairs of an object $[p] \in \Delta$ and a morphism $\sigma : \Delta[p] \rightarrow Z$, and morphisms those morphisms in Δ that commute with the maps to Z . In particular any p -simplex σ has face inclusion morphisms $d_i\sigma \rightarrow \sigma$ for each $i = 0, \dots, p$.*

A local system (of k -linear chain complexes) on a simplicial set Z is a functor

$$\begin{aligned} \mathcal{L} : \text{Simp}(Z) &\rightarrow \text{Ch}(k) \\ \sigma &\mapsto \mathcal{L}_\sigma \end{aligned}$$

to the category of chain complexes of k -modules, possibly unbounded in both directions, sending all morphisms in $\text{Simp}(Z)$ to quasi-isomorphisms.

If \mathcal{L} is such a system, we may define cochains of Z with coefficients in \mathcal{L} . In order to avoid confusion, let us decide that all differentials *decrease* degrees, and

that “cochains” therefore tends to live in negative degrees. Let

$$C^*(Z; \mathcal{L}) = \prod_{\sigma \in \text{Simp}(Z)} \mathcal{L}_\sigma,$$

which a priori is a bigraded k -vector space (one grading from the dimension of σ , one from the grading in $\text{Ch}(k)$) with two commuting coboundary maps (one from the internal boundary maps of the \mathcal{L}_σ , one from the alternating sum of the maps $\delta_i^j : \mathcal{L}_{d_i \sigma} \rightarrow \mathcal{L}_\sigma$). By $C^*(Z; \mathcal{L})$ we shall usually mean the associated total complex.

Remark 4.35. *Alternatively, a local coefficient system could be defined as a functor from $\text{Simp}(Z)$ into the category of Hk -module spectra, sending all morphisms into weak equivalences. The Dold–Kan functor and its homotopy inverse functor Lc discussed in Section 4.3 translate back and forth between these definitions.*

Taking cochains of a local system then corresponds to taking homotopy limit: if \mathcal{L} is a local system of Hk -module spectra, then applying Lc to the homotopy limit of \mathcal{L} , which is naturally again an Hk -module spectrum, results in a chain complex which is canonically quasi-isomorphic to the cochains of the local system of k -linear chain complexes $\sigma \mapsto Lc(\mathcal{L}_\sigma)$.

Let us comment on the relationship between this notion of local system and the more classical one as a module with an action of the fundamental group. In fact the classical one is sufficient for local systems of chain complexes whose homology is concentrated in a single degree, as follows.

Remark 4.36. *In the case where the base Z is path connected and $z \in Z$ is a vertex, the group $\pi_1(Z, z)$ acts on the homology of the chain complex \mathcal{L}_z . If $H_*(\mathcal{L}_z)$ is concentrated in a single degree, then this action in fact classifies the local system: any $\pi_1(Z, z)$ -module may be realized as $H_n(\mathcal{L}_z)$ for a local system \mathcal{L} such that \mathcal{L}_z has homology concentrated in degree n , and if \mathcal{L} and \mathcal{L}' are two local systems such that both \mathcal{L}_z and \mathcal{L}'_z have homology concentrated in degree n then any isomorphism $H_n(\mathcal{L}_z) \cong H_n(\mathcal{L}'_z)$ may be realized by a zig-zag of weak equivalences of functors $\text{Simp}(Z) \rightarrow \text{Ch}(k)$.*

Let us return to the tangent complex. Assume that $\mathcal{F} : \text{Art}_k \rightarrow s\text{Sets}/Z$ preserves homotopy pullbacks and that $\mathcal{F}(k) \in s\text{Sets}/Z$ is a terminal object. For each $\sigma : \Delta[p] \rightarrow Z \cong \mathcal{F}(k)$ we may define a functor $\mathcal{F}_\sigma : \text{Art}_k$ by the homotopy pullback square

$$\begin{array}{ccc} \mathcal{F}_\sigma(A) & \longrightarrow & \mathcal{F}(A) \\ \downarrow & & \downarrow \\ \Delta[p] & \xrightarrow{\sigma} & \mathcal{F}(k). \end{array}$$

Then each \mathcal{F}_σ has a tangent complex $\mathfrak{t}\mathcal{F}_\sigma$, and we have obtained a functor

$$\begin{aligned} \text{Simp}(Z) &\rightarrow \text{Ch}(k) \\ \sigma &\mapsto \mathfrak{t}\mathcal{F}_\sigma \end{aligned}$$

sending all morphisms to weak equivalences.

Definition 4.37. *Let Z be a simplicial set and $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}_{/Z}$ a homotopy invariant functor preserving pullbacks, such that $\mathcal{F}(k)$ is a terminal object. Then the tangent complex of \mathcal{F} is the local system of k -linear unbounded chain complexes on Z given by $\sigma \mapsto \mathfrak{t}\mathcal{F}_\sigma$, as defined above.*

Our results and constructions from Section 4.5 all have a generalizations to non-reduced functors, where the tangent complex is a local system on $\mathcal{F}(k)$ as in Definition 4.37. Most importantly for the next section, we have the following construction.

Example 4.38. *Let $\mathcal{F} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ be homotopy invariant and preserve homotopy pullbacks. Without loss of generality, suppose also that all values of \mathcal{F} are Kan. If we let $Z = \mathcal{F}(k)$ and regard \mathcal{F} as a functor $\mathbf{Art}_k \rightarrow \mathbf{sSets}_{/Z}$ the value at k is then terminal. Let $\mathfrak{t}\mathcal{F} : \mathbf{Simp}(Z) \rightarrow \mathbf{Ch}(k)$ be its tangent complex, regarded as a local system of k -linear chain complexes on Z , as defined above.*

Let X be any simplicial set and $\bar{\rho} : X \rightarrow Z = \mathcal{F}(k)$ a map. We may then define a new functor $\mathcal{F}_{X,\bar{\rho}} : \mathbf{Art}_k \rightarrow \mathbf{sSets}$ by

$$\mathcal{F}_{X,\bar{\rho}}(A) = \mathrm{hofib}_{\bar{\rho}}(\mathbf{sSets}(X, \mathcal{F}(A)) \rightarrow \mathbf{sSets}(X, \mathcal{F}(k))).$$

Then $\mathcal{F}_{X,\bar{\rho}}$ is formally cohesive, and its tangent complex is given by the formula

$$(4.15) \quad \mathfrak{t}\mathcal{F}_{X,\bar{\rho}} \simeq C^*(X, \bar{\rho}^* \mathfrak{t}\mathcal{F}),$$

where $\bar{\rho}^ \mathfrak{t}\mathcal{F}$ denotes the pulled-back local system defined by precomposing $\mathfrak{t}\mathcal{F}$ by $\bar{\rho}_* : \mathbf{Simp}(X) \rightarrow \mathbf{Simp}(Z)$.*

The formula (4.15) is perhaps best explained before turning Hk modules into chain complexes, where the definition of $\mathfrak{t}\mathcal{F}$ is more explicit. In that setting, a local system on X is simply a functor from the category of simplices of X to the category of Hk -module spectra, and taking cochains translates to a particular model for the homotopy limit. For the particular local system relevant for (4.15), the n th space of the functor sends a simplex $\sigma : \Delta^p \rightarrow X$ to the space of homotopy commutative diagrams

$$\begin{array}{ccc} \Delta^p & \longrightarrow & \mathcal{F}(k \oplus k[n]) \\ \sigma \downarrow & & \downarrow \\ X & \xrightarrow{\bar{\rho}} & \mathcal{F}(k), \end{array}$$

where “space of homotopy commutative diagrams” means the simplicial set of ways to supply the top horizontal arrow as well as a simplicial homotopy $\Delta^1 \times \Delta^p \rightarrow \mathcal{F}(k)$ in the diagram. Taking homotopy limit over all σ then gives the space

of homotopy commutative diagrams

$$\begin{array}{ccc} & & \mathcal{F}(k \oplus k[n]) \\ & \nearrow & \downarrow \\ X & \xrightarrow{\bar{p}} & \mathcal{F}(k), \end{array}$$

which is the n th space of Hk -module spectrum $\mathfrak{t}\mathcal{F}_{X, \bar{p}}$.

For later use, we remark that the construction in the above example works just as well when $X = (i \mapsto X_i)$ is a pro-simplicial set. In that case the example applies to each X_i and we may pass to the (filtered) colimit.

Finally, let us describe how to define $BGL_n(A)$, for A a simplicial ring. The definition we give is *ad hoc* and depends on the specific realization of $GL_n(A)$:

Example 4.39. For a simplicial ring A , let $GL_n(A) \subset M_n(A) = A^{n^2}$ denote union of the path components in $GL_n(\pi_0 A) \subset M_n(\pi_0 A)$. Then the functor $A \mapsto BGL_n(A)$ is homotopy invariant and preserves homotopy pullbacks, but $BGL_n(k)$ is not contractible, since it is a $K(\pi, 1)$ for $GL_n(k)$.

To see that BGL_n preserves homotopy pullback, we use Corollary 4.32. Indeed, the looped functor $A \mapsto \Omega BGL_n(A)$ is just $GL_n(A) \subset M_n(A) = A^{n^2}$. This looped functor $A \mapsto M_n(A)$ clearly preserves homotopy pullback, so it remains to see that $\pi_0(GL_n)$ preserves surjections of Artin rings. This is because a square matrix with coefficients in an Artin local ring is invertible if and only if the reduction to a matrix with coefficients in the residue field is invertible.

Lemma 4.40. The tangent complex of $A \mapsto BGL_n(A)$ is the local system on $BGL_n(k)$ given by the conjugation action of $GL_n(k)$ on $M_n(k)$. (Here $M_n(k)$ is discrete, i.e. the homotopy groups of the tangent complex vanish except in degree 0.)

We omit the proof, because in the next section we shall give a more general result: we define a functor $A \mapsto BG(A)$ for an arbitrary algebraic group G , and will prove the analogue of the Lemma in that context. (The definition of the next section will not specialize to Example 4.39 in the case $G = GL_n$, but it is naturally weakly equivalent to it.)

5. REPRESENTATION FUNCTORS

In this section we define and study an “infinitesimal representation variety” functor, parametrizing representations into an algebraic group G defined over $W(k)$. Assume for the moment that G has trivial center. Let Γ be a group. We wish to define a functor

$$(5.1) \quad A \mapsto \mathcal{F}_G(A) = \text{Hom}(\Gamma, G(A))/G(A),$$

where $G(A)$ acts by conjugation on the space of homomorphisms. Better yet, we want to study the subfunctor where the composition $\Gamma \rightarrow G(A) \rightarrow G(k)$ is fixed

to be some $\overline{\rho}$ on which $G(k)$ acts without isotropy. This makes perfect sense as stated for a discrete (pro-)group Γ and a discrete commutative $W(k)$ -algebra A , and for a homomorphism $A \rightarrow k$. In this section we will define such a functor also for simplicial A , and possibly (pro-)simplicial Γ as well.

The final definition of our representation functors are given in Definition 5.4 – the role of Γ is replaced by $B\Gamma$, which is also allowed to be in fact an arbitrary pro-simplicial set, as we need in our applications; and their tangent complexes are computed in Lemma 5.10.

5.1. $BG(A)$ for a simplicial ring A . Let us first discuss how one might try to define $G(A)$, if A is a simplicial ring. The simplicial group $[p] \mapsto G(A_p)$ is not a useful object to consider, even for $G = \mathbf{G}_m$: it is not homotopy invariant.

If \mathcal{O}_G is the (discrete) $W(k)$ -algebra with $G = \mathrm{Spec}(\mathcal{O}_G)$, we may instead define $A \mapsto G(A)$ for $A \in \mathrm{Art}_k$ by

$$G(A) = \mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_G), A),$$

where $c : \mathrm{SCR} \rightarrow \mathrm{SCR}$ denotes cofibrant approximation. This is a reasonable definition if we only care about G as a scheme, and not a group scheme. In fact, $G : \mathrm{Art}_k \rightarrow \mathrm{sSets}$ is pro-representable in the sense of Definition 2.19, e.g. by inverse system given by the objects $c(\mathcal{O}_G/p^m)$. Unfortunately, $A \mapsto G(A)$ defined this way does not take values in simplicial groups or even simplicial monoids.

For $G = \mathrm{GL}_n$ there is an *ad hoc* definition of $\mathrm{GL}_n(A)$ as the union of the components of $M_n(A) = A^{n^2}$ with invertible image in $M_n(\pi_0 A)$. This is monoid valued and weakly equivalent to the abstract definition above.

For a general algebraic group G , it seems better to directly define the functor $A \mapsto BG(A)$ instead of first defining $G(A)$. (If desired, a simplicial group valued functor $A \mapsto G(A)$ can be obtained by applying “Kan loop group”.)

For an ordinary commutative ring A , the space $BG(A)$ is the realization of the nerve of $G(A)$; the functor $A \mapsto N_p(G(A))$ (i.e. p -simplexes of the nerve of $G(A)$) is represented by $\mathcal{O}_{N_p G} = \mathcal{O}_G^{\otimes p}$, the p -fold tensor product over $W(k)$. Our recipe in general is to replace each $A \mapsto N_p G(A)$ by a functor which is representable by a simplicial ring which is cofibrant and homotopy discrete, in a way that also preserves the simplicial structure $[p] \mapsto N_p G(A)$.

In the simplicial case we define for $A \in \mathrm{Art}_k$ for each p the simplicial set

$$\mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_{N_p G}), A).$$

This simplicial set is a functor of $[p]$ as well as A , and we have obtained a functor from simplicial rings to bisimplicial sets.

Definition 5.1. *Let $BG(A)$ denote Ex^∞ of the geometric realization (i.e. the diagonal) of the bisimplicial set $\mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_{N_\bullet G}), A)$ defined above.*

Note that, if A is homotopy discrete, then $BG(A)$ is weakly equivalent to the usual classifying space of the discrete group $G(\pi_0 A)$. Let us also note the following “sanity check” of our definition.

Lemma 5.2. *Let $G(A) = \mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_G), A)$ as above. Then the natural inclusion*

$$G(A) \rightarrow \Omega BG(A)$$

is a weak equivalence.

Proof. For $R \in \mathrm{SCR}$, we say that $\mathbb{Z} \rightarrow R$ is “cellular” if it is isomorphic to a possibly transfinite) composition of cell attachments as in Definition 2.1.

Let us also note that $(c(\mathcal{O}_G))^{\otimes p}$ is cellular if $c(\mathcal{O}_G)$ is and hence for purposes where the strict functoriality in $[p] \in \Delta$ is unnecessary we may use $(c(\mathcal{O}_G))^{\otimes p}$ as a model for $c(\mathcal{O}_{N_p G})$. In particular we see that the morphisms $[1] \cong (i-1 < i) \subset (0 < \dots < p)$ in Δ for $i = 1, \dots, p$ induce weak equivalences

$$\mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_{N_p G}), A) \rightarrow \prod_{i=1}^p \mathrm{Hom}_{\mathrm{SCR}}(c(\mathcal{O}_{N_1 G}), A).$$

This condition on a simplicial space, often called the “Segal condition” after [30], implies that the natural map from the 1-simplices to the loop space of the geometric realization is a weak equivalence. \square

Corollary 5.3. *If $A \rightarrow B$ is a morphism in Art_k whose underlying map of simplicial sets is n -connected, then $BG(A) \rightarrow BG(B)$ is $(n+1)$ -connected.*

The proof of this Corollary is not hard. Before giving it, let us point out that we defined $G(A) = \mathrm{SCR}(c(\mathcal{O}_G), A)$ and not $\mathrm{SCR}_{/k}(c(\mathcal{O}_G), A)$. Hence we have an interesting map $G(A) \rightarrow G(k)$. Each element $\rho \in G(k) = \pi_0 G(k)$ gives a homomorphism $\rho : \mathcal{O}_G \rightarrow k$ and hence an object $(\mathcal{O}_G, \rho) \in \mathrm{SCR}_{/k}$, and we have an isomorphism of simplicial sets

$$(5.2) \quad G(A) \cong \coprod_{\rho \in G(k)} \mathrm{SCR}_{/k}((c(\mathcal{O}_G), \rho), A)$$

which is in fact a natural isomorphism of functors $\mathrm{Art}_k \rightarrow s\mathrm{Sets}$.

Proof. By Lemma 5.2 it suffices to see that $G(A) \rightarrow G(B)$ is n -connected, and by (5.2) it suffices to see that

$$\mathrm{SCR}_{/k}((c(\mathcal{O}_G), \rho), A) \rightarrow \mathrm{SCR}_{/k}((c(\mathcal{O}_G), \rho), B)$$

is n -connected for all $\rho \in G(k)$, which we may check one ρ at a time.

This holds because the coordinate ring \mathcal{O}_G is smooth at the k -point $\rho \in G(k)$. Hence if we choose formal parameters $x_1, \dots, x_d \in \mathcal{O}_G$ of G at ρ , the object of $\mathrm{pro}\text{-}\mathrm{Art}_k$ representing the functor $\mathrm{SCR}_{/k}((c(\mathcal{O}_G), \rho), -)$ may be taken to be $W(k)[[x_1, \dots, x_d]]$ for $d = \dim(G)$. This choice of parameters induce a natural weak equivalence $\mathrm{SCR}_{/k}((c(\mathcal{O}_G), \rho), A) \simeq (\mathfrak{m}(A))^d$. \square

Our definition of $A \mapsto BG(A)$ is also functorial in the algebraic group G .

5.2. Framed and unframed deformation functors. We can now define analogues of the framed and unframed representation functors for general G . Let us postpone a discussion of irreducibility (hence our functors need not be representable in general – they are “derived stacks”).

Before we proceed to the general definitions, we give first the simplest case. Let \mathcal{X} be a scheme – in our applications, the spectrum of a ring of integers in a number field – and let y_0 a geometric basepoint. If $A \in \text{Art}_k$ is discrete, and so $G(A)$ is simply a discrete group, then the set of conjugacy classes of representations $\pi_1(\mathcal{X}, x_0) \rightarrow G(A)$ are in correspondence with the étale torsors over \mathcal{X} with structure group $G(A)$. In turn, these correspond to homotopy classes of maps from the étale homotopy type X of \mathcal{X} to $BG(A)$. Here the étale homotopy type is understood to be the pro-simplicial set X that is associated to \mathcal{X} by Friedlander [10]². This motivates the basic version of our representation functor, which appears in (5.4) below.

In terms of the discussion above, part (i) of the following definition corresponds to keeping track of representations together with a basis for the underlying module (this is usually called “framed deformation rings” in number theory) and part (ii) corresponds to studying only lifts of a fixed representation $\pi_1(\mathcal{X}_0, x) \rightarrow G(k)$. The most important case for us will be the functor $\mathcal{F}_{X, \bar{\rho}}$ from (iii), in the case when X is the étale homotopy type of the ring of integers in a number field.

In the following definition, if $X = (\alpha \mapsto X_\alpha)$ is a pro-object in simplicial sets, we write $\text{Hom}(X, -)$ for the colimit

$$(5.3) \quad \text{colim}_\alpha \text{sSets}(X_\alpha, -).$$

Definition 5.4. (i) Let X be a pro-object in based simplicial sets, and let $\mathcal{F}_{X, G}^\square$ be the functor whose value on a simplicial ring A is the space of based maps $X \rightarrow BG(A)$. Equivalently, it is the homotopy fiber of the

$$\text{Hom}(X, BG(A)) \rightarrow BG(A),$$

given by evaluation at the basepoint.

(ii) Let X be a pro-object in simplicial sets, and let $\mathcal{F}_{X, G}$ be the functor whose value on a simplicial ring A is the space of maps (now unbased):

$$(5.4) \quad \mathcal{F}_{X, G}(A) = \text{Hom}(X, BG(A)).$$

(iii) If $\bar{\rho} : \pi_1 X \rightarrow G(k)$ is a homomorphism (from the fundamental pro-groupoid, in the unpointed situation), we define functors $\mathcal{F}_{X, \bar{\rho}}$ and $\mathcal{F}_{X, \bar{\rho}}^\square$ by taking homotopy fibers over the zero-simplices of $\mathcal{F}_{X, G}(k)$ and $\mathcal{F}_{X, G}^\square(k)$ given by $\bar{\rho}$.

It follows from the discussion of Example 4.38, that these functors are homotopy invariant and preserve homotopy pullback, and that the functors of (iii) are formally cohesive.

²In the cases of interest for number theory, we don’t need to appeal to the étale homotopy type – see the footnote on page 74 for a more straightforward construction.

5.3. Calculation of tangent complexes. The functor $\mathcal{F}_{X,G}$ is not reduced except in trivial cases. Hence the tangent complex is a local system (§4.7) on $\mathcal{F}_{X,G}(k)$, and in fact in Section 4.7 we have developed all the tools to calculate it at an arbitrary vertex $\bar{\rho} \in \mathcal{F}_{X,G}(k)$.

To this end, it is convenient to first treat the functor $A \mapsto BG(A)$, which we shall write as simply BG . It has the same properties as the functors defined above, namely it is homotopy invariant, Kan valued, preserves pullbacks, but is not reduced.

Lemma 5.5. *The tangent complex of $A \mapsto BG(A)$ is the local system on $BG(k)$ which at the basepoint gives a chain complex with homology concentrated in degree 1, where it is the k -module $\mathfrak{g} = \text{Ker}(G(k[\epsilon]/\epsilon^2) \rightarrow G(k))$ with the conjugation action of $G(k) = \pi_1(BG(k))$.*

In other words it is the Lie algebra of G with the adjoint action, concentrated in degree 1 (where as usual we insist on grading so that boundary maps decrease degrees; in cohomological notation it would be in degree -1). By Remark 4.36, the Lemma uniquely describes $t(BG)$ as a local system on $BG(k)$ up to weak equivalence of local systems.

Proof. Let us work with the corresponding spectra instead of chain complexes. Then the tangent complex $t(BG)$ at the basepoint vertex of $BG(k)$ is the spectrum whose n th space is

$$(5.5) \quad \text{hofib}(BG(k \oplus k[n]) \rightarrow BG(k)).$$

Now $BG(k \oplus k[n]) \rightarrow BG(k)$ is $(n+1)$ -connected by Corollary 5.3 and the $(n+1)$ st loop space is the homotopy fiber of $G(k \oplus k[0]) \rightarrow G(k)$ or in other words the kernel of that surjection of discrete groups. Hence the space (5.5) is an Eilenberg–MacLane space $K(\mathfrak{g}, n+1)$ for $\mathfrak{g} = \text{Ker}(G(k[\epsilon]/\epsilon^2) \rightarrow G(k))$, and this description also captures the action of $G(k)$. \square

From the Lemma and Example 4.38 it is then immediate to read off the tangent complex of associated functors, in particular:

Example 5.6. *With notation as in Definition 5.4, the tangent complex of $\mathcal{F}_{X,G,\bar{\rho}}$ is (quasi-isomorphic to) $C^{*+1}(X; \bar{\rho}^* \mathfrak{g})$, where we regard the Lie algebra \mathfrak{g} as a local system on X by means of $\bar{\rho}$.*

The indexing may be somewhat confusing: we mean that π_{-i} of the tangent complex is isomorphic to the $H^{i+1}(X; \bar{\rho}^* \mathfrak{g})$; in particular the homotopy groups of the tangent complex are concentrated in degrees $(-\infty, 1]$.

Observe, in particular, that if the center of G is positive-dimensional, then the functor $\mathcal{F}_{X,G,\bar{\rho}}$ is never pro-representable, because its tangent complex has nonvanishing π_1 . We now explain how to modify the foregoing discussion in this case.

5.4. Modifications for center. If G has a nontrivial center (e.g. $G = \mathrm{GL}_n$) the functor $\mathcal{F}_{X,G}$ will never be pro-representable, because of automorphisms. We now explain how to modify the previous discussion, along the lines of (5.1). (This section will not be used in the remainder of the paper.)

Suppose, then, that $Z \rightarrow G$ is a central algebraic subgroup. Then we have an algebraic group $PG = G/Z$ and hence $BPG(A)$ for simplicial A . It shall be convenient to give another model for $BPG(A)$, in which the short exact sequence $Z(A) \rightarrow G(A) \rightarrow PG(A)$ corresponds to a fibration sequence

$$BG(A) \rightarrow BPG(A) \rightarrow B^2Z(A).$$

We must first explain the definition of the functor $A \mapsto B^2Z(A)$. For discrete rings A , $Z(A)$ is a commutative group and hence $N_\bullet Z(A)$ is a simplicial group so we may form the bisimplicial set $N_\bullet N_\bullet Z(A)$. Extend this to a functor on simplicial rings which in each bi-degree is representable by a cofibrant simplicial ring, as before. Then $B^2Z(A)$ is defined as Ex^∞ of the diagonal of the trisimplicial set.

Lemma 5.7. *In the above setting, there is a natural fibration sequence*

$$BG(A) \rightarrow BPG(A) \rightarrow B^2Z(A).$$

Proof sketch/Definition. What we mean is that the functors are related by zig-zags of natural weak equivalences to functors which actually form a fibration.

The main thing to explain is how to define $BPG(A) \rightarrow B^2Z(A)$, to which end we replace $BPG(A)$ by the Borel construction $BG(A) // BZ(A)$. More explicitly, for discrete A the multiplication $Z(A) \times G(A) \rightarrow G(A)$ is a group homomorphism and hence gives an action $(N_p Z(A)) \times (N_p G(A)) \rightarrow N_p(G(A))$ of the simplicial group $N_\bullet Z(A)$ on the simplicial set $N_\bullet G(A)$. Taking bar construction of this actions gives a bisimplicial set

$$(p, q) \mapsto N_q(*, N_p Z(A), N_p G(A)),$$

which is represented by a bi-cosimplicial ring which in bidegree (p, q) is $\mathcal{O}_Z^{\otimes pq} \otimes \mathcal{O}_G^{\otimes p}$. Then apply a functorial cosimplicial replacement as before, and redefine $BPG(A)$ as the geometric realization of the tri-simplicial set represented (this is naturally weakly equivalent to the previous definition). There is a natural weak equivalence to the previously defined $BPG(A)$.

In this model the obvious map $N_q(*, N_p Z(A), N_p G(A)) \rightarrow N_q(*, N_p Z(A), *)$ gives the desired map $BPG(A) \rightarrow B^2Z(A)$, with fiber $BG(A)$. (Note that, as before, we should apply Ex^∞ to all these functors so that they have Kan values.) \square

Definition 5.8. *Let X be a pro-object in spaces, and let $\mathcal{F}_{X,G,Z}$ be defined by the homotopy pullback diagram*

$$(5.6) \quad \begin{array}{ccc} \mathcal{F}_{X,G,Z}(A) & \longrightarrow & \mathrm{Hom}(\pi_0 X, B^2Z(A)) \\ \downarrow & & \downarrow \mathrm{const} \\ \mathrm{Hom}(X, PG(A)) & \longrightarrow & \mathrm{Hom}(X, B^2Z(A)) \end{array}$$

where the right-hand vertical map is the inclusion of the (componentwise) constant maps; and $\mathrm{Hom}(X, -)$ is as in (5.3), but taken in unpointed spaces.

Observe that when Z is trivial, $\mathcal{F}_{X,G,Z}$ reduces to the functor $\mathcal{F}_{X,G}$ previously studied. More generally $\mathcal{F}_{X,G,Z}$ fits into a natural fibration sequence

$$(5.7) \quad \mathrm{Hom}(\pi_0(X), BZ(A)) \rightarrow \mathrm{Hom}(X, BG(A)) \rightarrow \mathcal{F}_{X,G,Z}(A).$$

In the case $X = B\Gamma$ for a pro-simplicial group Γ , this could be thought of as modeling a bundle of stacks

$$*\!/Z \rightarrow (\mathrm{Hom}(\Gamma, G))\!/G \rightarrow (\mathrm{Hom}(\Gamma, G))\!/PG.$$

Lemma 5.9. *$\mathcal{F}_{X,G,Z}$ is homotopy invariant and preserves homotopy pullbacks.*

Proof. The three functors in the homotopy pullback diagram defining $\mathcal{F}_{X,G,Z}$ are all special cases of the construction in Example 4.38, hence they are all homotopy invariant and preserve homotopy pullback. These properties are then inherited by $\mathcal{F}_{X,G,Z}$. \square

If $\bar{\rho} : \pi_1 X \rightarrow G(k)$ is a homomorphism, we may define functors $\mathcal{F}_{X,Z,\bar{\rho}}$ and $\mathcal{F}_{X,Z,\bar{\rho}}^\square$ similarly to the previous discussion. We may compute the tangent complexes of $\mathcal{F}_{X,G,Z}$ and $\mathcal{F}_{X,Z,\bar{\rho}}$: the fiber sequence (5.7) induces a cofiber sequence of tangent complexes and the tangent complexes of $\mathrm{Hom}(\pi_0(X), BZ(A))$ and $\mathrm{Hom}(X, BG(A))$ are determined by the above discussion. We deduce a cofiber sequence

$$C^{*+1}(\pi_0 X; \mathrm{Ad}(Z)) \rightarrow C^{*+1}(X; \mathrm{Ad}(G)) \rightarrow \mathbf{t}\mathcal{F}_{X,Z,\bar{\rho}},$$

where we have written $\mathrm{Ad}(G)$ for the Lie algebra of G with action of $\pi_1(X)$ pulled back along $\bar{\rho}$ and similarly for $\mathrm{Ad}(Z)$.

Lemma 5.10. *The tangent complex of $\mathcal{F}_{X,Z,\bar{\rho}}$ is quasi-isomorphic to the mapping cone of*

$$C^{*+1}(\pi_0(X); \mathrm{Ad}(Z)) \rightarrow C^{*+1}(X; \mathrm{Ad}(G)),$$

where $\mathrm{Ad}(G)$ denotes the Lie algebra of G over k , acted on by $\pi_1(X)$ via $\bar{\rho}$, and C^* denotes cochains. The map is induced by the inclusion $Z \subset G$ and by the canonical map $X \rightarrow \pi_0(X)$ of (pro-)simplicial sets, and the domain is non-vanishing only for $* = -1$.

In particular, the tangent complex is co-connective precisely when the map of k -vector spaces $H^0(\pi_0(X); \mathrm{Ad}(Z)) \rightarrow H^0(X; \mathrm{Ad}(G))$ is an isomorphism, i.e. only central vectors in the Lie algebra of $G(k)$ are fixed by the action of the pro-groupoid $\pi_1(X)$. In that case we have an isomorphism

$$\pi_{-n} \mathbf{t}\mathcal{F}_{X,Z,\bar{\rho}} = H^{n+1}(X; \mathrm{Ad}(G))$$

for all $n \geq 0$.

Proof. This follows from the discussion in Section 4.5, together with the fiber sequence (5.7). \square

Definition 5.11. A homomorphism $\bar{\rho} : \pi_1 X \rightarrow G(k)$ is irreducible if the map of k -vector spaces $H^0(\pi_0 X; \text{Ad}(Z)) \rightarrow H^0(X, \text{Ad}(G))$ is an isomorphism.

For $G = \text{GL}_n$ this notion of irreducibility is sometimes called *absolute* irreducibility: it amounts to the representation of $\pi_1 X$ being irreducible in the sense of representation theory, even after extending scalars to an algebraic closure of k .

6. NUMBER-THEORETIC NOTATION: GALOIS REPRESENTATIONS

The remainder of this paper is devoted to studying the derived deformation ring of a Galois representation. Both this section and the next one (§7) set up notation that we will use. The current section collects notation related to Galois representations and their cohomology, while §7 collects notation related to deformation rings and problems.

6.1. General setup. In our number theory sections, we shall fix a prime p and a finite field k of characteristic p with Witt vectors $W(k)$. (In the prior sections we have often used p to denote “simplicial degree,” but we will never do so hereafter.) We let $W_n = W(k)/p^n$ be the corresponding ring of length n Witt vectors.

Let S be a finite set of primes of \mathbb{Q} containing p . Let $T = S - \{p\}$.

Let G be a split semisimple algebraic group over $W(k)$, e.g. $G = \text{PGL}_n$. This will be the target for our Galois representations. To avoid various (not particularly important) subtleties with square roots, it is convenient for us to restrict to the case when G is actually adjoint.³

Let $T \subset G$ be a maximal k -split torus, and let the Lie algebras of T, G be denoted by $\text{Lie}(T), \text{Lie}(G)$ respectively; these are free $W(k)$ -modules, and we denote $\text{Lie}(T)_k = \text{Lie}(T) \otimes_{W(k)} k$ and $\text{Lie}(G)_k = \text{Lie}(G) \otimes_{W(k)} k$. Where typographically convenient we will use the shorthand \mathfrak{g} for $\text{Lie}(G)_k$; we don’t use this shorthand for T because it clashes with notation for the tangent complex.

Since we will be passing often between simplicial rings and usual rings, in these sections we will adhere to the following convention: script letters \mathcal{A}, \mathcal{B} etc. denote simplicial commutative rings, and roman letters A, B etc. always denote usual rings. (A normally italicized letter R could denote either.)

6.2. Galois groups. With S as above, a finite set of primes containing p , we put

$$\Gamma_S := \pi_1^{\text{et}}(\mathbb{Z}[\frac{1}{S}], x_0) \cong \text{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q})$$

where x_0 is a fixed basepoint, and $\mathbb{Q}^{(S)}$ is the largest extension of \mathbb{Q} unramified outside S .

³It might be worth remarking here that in, for example, the theory of modular forms for GL_2 one usually studies Galois representations to GL_2 with fixed determinant. However, this deformation functor maps to the corresponding PGL_2 deformation functor, and, away from characteristic 2, the map is an isomorphism by inspection of tangent spaces. Thus in such situations, so long as the characteristic does not divide the order of $\pi_1(G)$, nothing is lost by working with the adjoint form.

It is traditional in number theory to think in terms of Galois groups, but given that our definition of representation rings is actually specified in terms of maps out of the étale homotopy type, it seems more consistent for us to use π_1 instead. We will usually be lazy about specifying basepoints, just as one is often lazy about algebraic closures when discussing Galois groups. (To be precise, we should fix, once and for all, basepoints for the spectrum of $\mathbb{Z}[\frac{1}{S}]$, \mathbb{Z}_q and the other rings we use; no compatibility between these basepoints is required.)

For every finite prime q there is a distinguished conjugacy class of mappings

$$(6.1) \quad \iota_q : \pi_1(\mathbb{Q}_q, x_1) \longrightarrow \Gamma_S$$

where again x_1 is a fixed basepoint; we have $\pi_1(\mathbb{Q}_q, x_1) \cong \text{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$. Moreover for $q \notin S$ the map ι_q factors through the unramified quotient of $\pi_1(\mathbb{Q}_q, x_1)$. This unramified quotient is pro-cyclic, generated by a Frobenius element at q :

$$\pi_1(\mathbb{F}_q, x_2) \cong \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \langle \text{Frob}_q \rangle.$$

The abelianization $\pi_1 \mathbb{Q}_q^{\text{ab}}$ of $\pi_1 \mathbb{Q}_q$ is isomorphic, by class field theory, to the profinite completion of \mathbb{Q}_q^* . The tame part $\pi_1 \mathbb{Q}_q^{\text{ab, tame}}$ of this abelianization is, by definition, the quotient by the maximal pro- q subgroup $1 + q\mathbb{Z}_q$ of \mathbb{Z}_q^* . Thus it fits into a short exact sequence

$$(6.2) \quad \underbrace{I_q}_{\cong (\mathbb{Z}/q)^*} \hookrightarrow \pi_1 \mathbb{Q}_q^{\text{ab, tame}} \twoheadrightarrow \underbrace{\pi_1 \mathbb{F}_q}_{\cong \hat{\mathbb{Z}}},$$

where we have defined I_q to be the kernel of the natural surjection on the right. The notation I_q here is intended to suggest “inertia,” but it is only a small quotient of the full inertia group: it is just the tame part of the abelian Galois group.

For any p -torsion abelian group M equipped with a Γ_S -action, the natural map

$$H^*(\Gamma_S, M) \xrightarrow{\sim} H_{\text{et}}^*(\mathbb{Z}[\frac{1}{S}], M)$$

is an isomorphism, and we will denote this group by $H^*(\mathbb{Z}[\frac{1}{S}], -)$ where convenient, without explicitly writing the subscript “etale.” Similarly for $H^*(\mathbb{Q}_q, -)$ and $H^*(\mathbb{F}_q, -)$.

6.3. Cohomology with local conditions. See §B for details:

Let M be a p -torsion module under Γ_S . Let $S' \subset S$. For each $v \in S'$ suppose given a subgroup $\mathfrak{l}_v \subset H^1(\mathbb{Q}_v, M)$. One defines $H_{\mathfrak{l}}^1(\mathbb{Z}[\frac{1}{S}], M)$ (“cohomology with local conditions”) to be the subgroup of $x \in H^1(\mathbb{Z}[\frac{1}{S}], M)$ such that the restriction x_v to $H^1(\mathbb{Q}_v, M)$ lies in \mathfrak{l}_v for every $v \in S'$. There is an exact sequence

$$(6.3) \quad 0 \rightarrow H_{\mathfrak{l}}^1(\mathbb{Z}[\frac{1}{S}], M) \rightarrow H^1(\mathbb{Z}[\frac{1}{S}], M) \rightarrow \prod_{v \in S'} H^1(\mathbb{Q}_v, M)/\mathfrak{l}_v$$

$$(6.4) \quad \rightarrow H_{\mathfrak{l}}^2(\mathbb{Z}[\frac{1}{S}], M) \rightarrow H^2(\mathbb{Z}[\frac{1}{S}], M) \rightarrow \prod_{v \in S'} H^2(\mathbb{Q}_v, M) \rightarrow .$$

where one defines $H_{\mathfrak{l}}^2(\mathbb{Z}[\frac{1}{S}], M)$ by a cone construction (see §B or [7] – there is a specific lift of \mathfrak{l} to the chain level being used here, prescribed in §B.4).

If $S' = S$ it follows from duality in Galois cohomology (see §B) that there is a perfect pairing

$$H_{\mathfrak{l}}^1(M) \times H_{\mathfrak{l}^\perp}^2(M^*) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where M^* is the dual group of homomorphisms $M \rightarrow \mu_{p^\infty}$ and \mathfrak{l}^\perp is the dual local condition, i.e. \mathfrak{l}_v^\perp is the orthogonal complement to $H_{\mathfrak{l}_v}^1$ inside $H^1(\mathbb{Q}_v, M^*)$, with respect to the pairing of local Tate duality.

6.4. Fontaine–Laffaille and the f cohomology. We briefly recall the theory of Fontaine and Laffaille. Fix once and for all an interval $[a, b] \subset \mathbb{Z}$ of “Hodge weights”, where $b - a \leq p - 2$; for us, the interval $[-\frac{p-3}{2}, \frac{p-3}{2}]$ will do nicely. We say that a representation of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with coefficients in a finitely generated \mathbb{Z}_p -module M is *crystalline* if it is isomorphic to the quotient L_1/L_2 , where L_1, L_2 are lattices inside a crystalline representation $L_1 \otimes \mathbb{Q}_p = L_2 \otimes \mathbb{Q}_p$ with weights in the interval $[a, b]$.

Fontaine and Laffaille describe an explicit category MF of semilinear algebraic data (the “Fontaine–Laffaille modules”). See [9] or for a summary [2, p. 363]. There is an equivalence of categories

$$MF \xrightarrow{FL} \text{crystalline Galois modules}$$

If M is a crystalline module, we define the “ f -cohomology” $H_f^1(\mathbb{Q}_p, M)$ to be that subset of the Galois cohomology group $H^1(\mathbb{Q}_p, M)$ consisting of classes classifying extensions $M \rightarrow \tilde{M} \rightarrow \mathbb{Z}_p$ with \tilde{M} crystalline. This is known to be a subgroup of $H^1(\mathbb{Q}_p, M)$.

Globally for M a module under $\pi_1 \mathbb{Z}[\frac{1}{S}]$ we define the global f -cohomology

$$(6.5) \quad H_f^*(\mathbb{Z}[\frac{1}{S}], M)$$

by imposing conditions \mathfrak{l} only at $v = \{p\}$, taking $\mathfrak{l}_p = H_f^1(\mathbb{Q}_p, M) \subset H^1(\mathbb{Q}_p, M)$.

Remark. Note global f -cohomology is often used to denote imposing, in addition to crystalline conditions at p , *unramified* conditions at all places except p . If we want to use this convention we will write instead $H_f^*(\mathbb{Q}, M)$. When we write $H_f^*(\mathbb{Z}[\frac{1}{S}], M)$ or similar we always mean that we impose *only* the crystalline condition at p , and impose no restrictions at $v \neq p$.

6.5. The arithmetic manifold $Y(K)$ and the existence of Galois representations. We recall below the definition of the arithmetic manifold $Y(K)$, associated to an algebraic \mathbb{Q} -group \mathbf{G} whose dual is G , and an open compact subgroup K of its adelic points.

Galois representations into the group G are related, by the Langlands program, to automorphic forms on a group *dual* to G . Therefore, let \mathbf{G} be the split reductive \mathbb{Q} -group whose root datum is dual to that of G ,⁴ and let $\mathbf{B} \supset \mathbf{T}$ be a Borel and

⁴We apologize for not introducing the customary “dualizing” signs, such as \vee , into the notation here: it seems too much of a conflict with standard notation to introduce use such notation for the

maximal split torus in \mathbf{G} . (Note that, to be more canonical, we could always replace \mathbf{T} in our considerations with the torus quotient of \mathbf{B}).

Fix a maximal compact subgroup K_∞° of the connected component of $\mathbf{G}(\mathbb{R})$. A “level structure” K for us is an open compact subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ of the points of \mathbf{G} over the finite adeles \mathbb{A}_f ; we will only consider examples where K is a product $K = \prod_q K_q$, with $K_q \subset \mathbf{G}(\mathbb{Q}_q)$, and where the “level is prime to p ” – we require that K_p is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Z}_p)$ (this means: the \mathbb{Z}_p -points of a smooth reductive model over \mathbb{Z}_p).

For such a level structure K , define the “arithmetic manifold of level K ”:

$$Y(K) = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_\infty^\circ K.$$

In fact, we can identify $Y(K)$ with a finite union of quotients of the (contractible) symmetric space \mathcal{Y} for $\mathbf{G}(\mathbb{R})$, by various arithmetic subgroups $\Gamma_i \leq \mathbf{G}(\mathbb{Q})$:

$$Y(K) = \coprod_i \Gamma_i \backslash \mathcal{Y}.$$

In particular, the orbifold cohomology of $Y(K)$ is the direct sum $\bigoplus_i H^*(\Gamma_i)$. It is one candidate for the “space of modular forms for \mathbf{G} .” In particular, the Langlands program predicts that, to a Hecke eigenclass on $H^*(Y(K), -)$, there should be attached a Galois representation into G . We now formulate this prediction precisely, in the form of the conjecture below.

6.6. We now formulate more precisely the conjecture on Galois representations to be used. We follow the version used by Khare–Thorne [20]; the version of Calegari–Geraghty does not use derived categories.

Let E denote a pro- p coefficient ring, with a map $E \rightarrow k$.

We consider pairs $(K \triangleleft K')$ of level structures, with $\Delta := K'/K$ abelian. Now Δ acts in the natural way on $Y(K)$, and we may consider $C_*^\Delta(Y(K), E)$, the chain complex of $Y(K)$ with E coefficients, as an object in the derived category of $E\Delta$ -modules. Each Hecke operator gives an endomorphism of this object. Let $\tilde{\mathbf{T}}_K$ be the ring of endomorphisms thus generated by all (prime-to-the-level) Hecke operators. It is commutative. Also by [20, Lemma 2.5] the natural map from $\tilde{\mathbf{T}}_K$ to the usual (homological) Hecke algebra has nilpotent kernel.

Conjecture 6.1. *Fix a surjection $\tilde{\mathbf{T}}_K \twoheadrightarrow k$ with kernel the maximal ideal \mathfrak{m} of $\tilde{\mathbf{T}}_K$. Let $\tilde{\mathbf{T}}_{K,\mathfrak{m}}$ be the completion of $\tilde{\mathbf{T}}_K$ at the maximal ideal \mathfrak{m} .*

Then there exists a Galois representation $\bar{\sigma} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(k)$ with the following properties:

group on which the automorphic forms live; and it would introduce too much typographical difficulty to call our G instead G^\vee . Since \mathbf{G} is always used in boldface and comes up rather rarely, we hope this will not be too confusing for the reader.

- (a) For all primes $q \neq p$ at which K_q is hyperspecial, the representation $\bar{\sigma}$ is unramified; moreover, if we fix a representation τ of G , the trace $\text{trace}(\tau \circ \sigma)(\text{Frob}_q) \in k$ coincides with the image of the associated Hecke operator $T_{q,\tau} \in \tilde{T}_K$ under $\tilde{T}_K \rightarrow k$. (Explicitly, $T_{q,\tau}$ comes from the Satake isomorphism of the Hecke algebra at q with the representation ring of G).
- (b) $\bar{\sigma}$ is odd at ∞ , i.e., the image of complex conjugation in $G(k)$ may be lifted to an involution in $G(W(k))$ whose trace, acting on $\text{Lie}(G)$ in the adjoint action, is minimal amongst all involutions.
- (c) At p , let $\bar{\sigma}_p : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow G(k)$ be the restriction of $\bar{\sigma}$. Then $\text{Ad}(\bar{\sigma}_p) : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{GL}(\mathfrak{g})$ is torsion crystalline (see §6.4).

Moreover, if the image of $\bar{\sigma}$ is large enough, e.g. if

$$(6.6) \quad \text{image}(\bar{\sigma}) \supset \text{image}(G^{\text{sc}}(k) \rightarrow G(k)),$$

where G^{sc} is the simply connected cover of G , then σ can be lifted to a representation $\sigma : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(\tilde{T}_{K,m})$ which continues to have the obvious analogue of property (a), and moreover:

- (d) At places v for which $K_v \neq K'_v$, one has the “local-global compatibility” formulated later: see Assumption 2 in § 13.5. This assumption is in fact only formulated for specific pairs (K_v, K'_v) , and these are the only type for which we will apply it; we regard (d) as being vacuous in other cases.
- (e) Let $\text{Def}_{\bar{\sigma}_p}$ be the usual (Mazur) deformation functor, from usual Artin rings augmented over k , to sets. There exists a unobstructed subfunctor $\text{Def}_{\bar{\sigma}_p}^{\text{crys}} \subset \text{Def}_{\bar{\sigma}_p}$ (“unobstructed” means that square zero extensions $\tilde{A} \rightarrow A$ induce surjections $\text{Def}_{\bar{\sigma}_p}^{\text{crys}}(\tilde{A}) \twoheadrightarrow \text{Def}_{\bar{\sigma}_p}^{\text{crys}}(A)$) with tangent space $H_f^1(\mathbb{Q}_p, \text{Ad}\rho_p) \subset H^1(\mathbb{Q}_p, \text{Ad}\rho_p)$, such that

$$\sigma \in \text{Def}_{\bar{\sigma}_p}^{\text{crys}}(\tilde{T}_{K,m}).$$

Note that asking about \tilde{T} is a slightly stronger statement than asking about the usual Hecke algebra T . The idea of considering \tilde{T} is due to Khare and Thorne [20], and evidence for this stronger statement has been given by Newton and Thorne [26].

The necessity of condition (e) is that the notion of crystalline deformation has not been explicated for a general group, although it has been verified in some important cases (e.g. GL_n in [7], and the case of GSp is apparently analyzed in Patrikis’ undergraduate thesis). We just assume this as an axiom. Note also that our assumption (c) that the adjoint representation is crystalline forces p to be “large” relative to \mathbf{G} : one expects the Hodge weights of $\text{Ad}(\bar{\rho}_p)$ to lie in the interval $[1 - h, h - 1]$ where h is the Coxeter number of G .

6.7. Taylor–Wiles primes. We summarize briefly the idea of Taylor–Wiles primes, which will play a central role in our analysis. Let

$$\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$$

be a representation.

A *Taylor–Wiles prime* q , for the Galois representation ρ , is a prime $q \notin S$ with the property $q \equiv 1 \pmod{k}$ and $\rho(\text{Frob}_q)$ is conjugate to a *strongly regular* element of $T(k)$ – as usual, a strongly regular element $t \in T(k)$ is an element t whose centralizer in G coincides with T . Therefore, if $G = \text{PGL}_n$, a Taylor–Wiles prime is simply one for which the Frobenius has a representative in $\text{GL}_n(k)$ with distinct eigenvalues, all belonging to k .

In particular, if q is a Taylor–Wiles prime, one may choose a representation

$$(6.7) \quad \rho_{\mathbb{Q}_q}^T : \pi_1 \mathbb{Q}_q \rightarrow T(k)$$

factoring through $\pi_1 \mathbb{Z}_q$, such that the composition of $\rho_{\mathbb{Q}_q}^T$ with $T \hookrightarrow G$ is isomorphic to $\rho_{\mathbb{Q}_q}$. We denote by $\rho_{\mathbb{Z}_q}^T$ the corresponding representation of $\pi_1 \mathbb{Z}_q$. Later we will study the T -valued deformation theory of $\rho_{\mathbb{Q}_q}^T$. This depends on the choice of $\rho_{\mathbb{Q}_q}^T$; it is often convenient to think of a Taylor–Wiles prime as being equipped with such a choice. Thus our basic objects are actually pairs (q, t) , where $t \in T(k)$ is regular and conjugate to $\rho(\text{Frob}_q)$. That then fixes $\rho_{\mathbb{Q}_q}^T$, by requiring that $\rho_{\mathbb{Q}_q}^T$ carry a Frobenius element to t .

What is really important to us are certain sets of Taylor–Wiles primes:

Definition 6.2. Let $\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$. An allowable Taylor–Wiles datum of level n , for the Galois representation ρ , is a set of Taylor–Wiles primes $Q = \{\ell_1, \dots, \ell_k\}$, disjoint from S , and each equipped with a regular element $t_{\ell_i} \in T(k)$ conjugate to $\rho(\text{Frob}_{\ell_i})$, and satisfying the further conditions:

- (a) p^n divides each $\ell_i - 1$
- (b) We have $H_1^2(\mathbb{Z}[\frac{1}{SQ}], \text{Ad}\rho) = 0$, where we impose (see §6.3, §6.4) local

$$\text{conditions } \mathfrak{l}_v \text{ at } v \in S \cup Q, \text{ namely } \mathfrak{l}_v = \begin{cases} H_f^1, & v = p \\ H^1, & v \in Q \\ 0, & v \in S - p. \end{cases}.$$

Note that although the choice of elements of $T(k)$ is part of the datum, we will often just informally say “let Q be a Taylor–Wiles datum,” with the understanding that one actually carries along this extra information.

More explicitly, the vanishing condition in (b) means that in the sequence (we write SQ instead of $S \coprod Q$):

$$(6.8) \quad \begin{aligned} H^1(\mathbb{Z}[\frac{1}{SQ}], \text{Ad}\rho) &\xrightarrow{A} \frac{H^1(\mathbb{Q}_p, \text{Ad}\rho)}{H_f^1(\mathbb{Q}_p, \text{Ad}\rho)} \oplus \bigoplus_{v \in S - \{p\}} H^1(\mathbb{Q}_v, \text{Ad}\rho) \\ &\rightarrow \underbrace{H_1^2}_0 \rightarrow H^2(\mathbb{Z}[\frac{1}{SQ}], \text{Ad}\rho) \xrightarrow{B} \bigoplus_{SQ} H^2(\mathbb{Q}_v, \text{Ad}\rho), \end{aligned}$$

that A is surjective and B is injective. Actually, the cokernel of B is dual to $H^0(\text{Ad}^*\rho(1))$; this will vanish so long as ρ has sufficiently large image, so if B is injective it is an isomorphism.

Remark. One can find allowable Taylor–Wiles data of any level n as long as ρ satisfies a “big image” criterion; for example, it suffices to assume that ρ restricted to $\mathbb{Q}(\zeta_{p^\infty})$ has image that contains the image of $G^{\text{sc}}(k) \rightarrow G(k)$, where G^{sc} is the simply connected cover. One proves this as in [11, Proposition 5.9]. In particular, write E for the fixed field of the kernel of ρ and $E' = E(\zeta_{p^n})$. The main point of the proof is to check that the restriction map from the cohomology of $\mathbb{Z}[\frac{1}{S}]$, with $\text{Ad}^*\rho(1)$ coefficients, to the corresponding cohomology group for E' is injective. To see this, in turn, it is enough to verify that the cohomology of $G := \text{Gal}(E'/F)$ acting on $\text{Ad}^*\rho(1)$ is trivial. Let N be the $(\mathbb{Z}/p)^*$ subgroup naturally embedded in $\text{Gal}(E'/E)$; it is a normal subgroup of G , and the claim follows easily from the fact that N has no invariants on $\text{Ad}^*\rho(1)$.

7. DEFORMATION-THEORY NOTATION

We continue our summary of notation for the rest of the paper. Here we briefly repeat some facts of homotopical algebra from the first part and set up notation for deformation rings and deformation functors.

7.1. Representable functors. As discussed at length in §2 we will deal extensively with objects of the category

$$(7.1) \quad \mathcal{M} = \text{simplicially enriched functors } \text{Art}_k \longrightarrow s\text{Sets},$$

namely, the deformation functors associated to various Galois representations. (We will almost always want to work with functors that are valued in Kan simplicial sets.)

A morphism is a natural transformation of simplicially enriched functors. There is a natural notion of weak equivalence for morphisms, namely, an object-wise weak equivalence, i.e. $F \rightarrow G$ is a weak equivalence when $F(A) \rightarrow G(A)$ is a weak equivalence for all $A \in \text{Art}_k$. Indeed \mathcal{M} has a compatible model category structure (mentioned in [36, §2.3.1]) with object-wise fibrations but we won’t use it.

We will however use the following shorthand: Given objects $F, G \in \mathcal{M}$, we will write

$$F \dashrightarrow G$$

to abbreviate an explicit zig-zag of maps

$$(7.2) \quad F \xleftarrow{\sim} F_1 \rightarrow F_2 \xleftarrow{\sim} F_3 \rightarrow F_4 \cdots \rightarrow G,$$

where all left arrows are weak equivalences. *Wherever we use this notation we have in mind an explicit zig-zag, not merely an morphism in the homotopy category of \mathcal{M} .* In practice it would be overly cumbersome to write out this zig-zag in every case, thus our shorthand.

Recall that, for any such zig-zag, we can find a “roof”: a functor F^* equipped with a natural weak equivalence $F^* \xrightarrow{\sim} F$, and a map $F^* \rightarrow G$; we can replace the long-zig zag above by

$$(7.3) \quad F \xleftarrow{\sim} F^* \rightarrow G.$$

Explicitly, we can take $F^*(A)$ to be the homotopy limit of the diagram $F(A) \leftarrow F_1(A) \rightarrow F_2(A) \cdots$, that is to say, a collection of points $x \in F(A)$, $z_1 \in F_1(A)$, $z_2 \in F_2(A)$ etc., together with a collection of paths: a path from the image of z_1 to x , etc.

We will use homotopy fiber products of functors: for $F_1 \rightarrow F_2 \leftarrow F_3$ we denote by $F_1 \times_{F_2}^h F_3$ the functor that assigns to $A \in \text{Art}_k$ the homotopy pullback $F_1(A) \times_{F_2(A)}^h F_3(A)$. (The homotopy pullback is defined in Definition A.4: informally speaking, it consists of points $v_i \in F_i(A)$ together with paths in F_2 between v_2 and the images of v_1, v_3 .) We say that a square in \mathcal{M}

$$(7.4) \quad \begin{array}{ccc} G & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ F_3 & \longrightarrow & F_2 \end{array}$$

is a homotopy pullback square if the induced map $G \rightarrow F_1 \times_{F_2}^h F_3$ is an object-wise weak equivalence.

Recall (Definition 2.19) also that a functor is *pro-representable* if it is naturally weakly equivalent to a functor of the form

$$\varinjlim_{\alpha} \text{Hom}_{\text{Art}_k}(\mathcal{R}_{\alpha}, -)$$

where the \mathcal{R}_{α} are a projective system of cofibrant objects of Art_k , indexed by $\alpha \in \text{Ob}\mathcal{C}$ for some cofiltered category \mathcal{C} . (Each Hom space above is the simplicial set of maps in the category $\text{SCR}/_k$, i.e. those maps of simplicial commutative rings that commute with the map to k .)

In this case, the pro-ring $\alpha \mapsto \mathcal{R}_{\alpha}$ is said to represent the given functor (for more precision, see Lemma 2.20); we note here that such a representing pro-ring is not determined up to unique isomorphism, but it is unique in a homotopical sense (see Lemma 2.25 or (3.4)). We will nonetheless slightly abuse notation and refer to “the” representing ring for a functor. Note that although we often write “deformation ring” or “representing ring,” these are always permitted to be pro-objects of SCR .

We say (*loc. cit.*) that a functor is *sequentially pro-representable* if it is possible to choose the indexing category \mathcal{C} to be the natural numbers. By Corollary 4.5 this is automatically the case if the tangent complex of the functor is of countable dimension. *This will be the case in all the applications to number theory, and thus we will freely take our functors to be represented by pro-systems indexed by the natural numbers.*

The sequentially indexed projective system \mathcal{R}_n ($n \in \mathbb{N}$) is said to be *nice* when each \mathcal{R}_n is cofibrant and all the transition maps $\mathcal{R}_n \rightarrow \mathcal{R}_m$ are fibrations; this is a special case of Definition 2.23. Any sequentially pro-representable functor can be represented by a nice pro-ring (Lemma 2.22).

It will be therefore be convenient to use the following notation: for a pro-object of \mathbf{Art}_k , say $\mathcal{R} = (\mathcal{R}_\alpha)$, we understand the functor Hom to mean

$$(7.5) \quad \mathrm{Hom}(\mathcal{R}, -) := \mathrm{colim}_\alpha \mathrm{Hom}_{\mathbf{Art}_k}(\mathcal{R}_\alpha, -).$$

However, if \mathcal{R} is not level-wise cofibrant this definition is not homotopy invariant, and we should only use the functor after first applying a level-wise cofibrant replacement. We fix throughout a cofibrant replacement functor c on the category \mathbf{SCR} ; we will write it as $\mathcal{S} \mapsto c(\mathcal{S})$ or (more often) $\mathcal{S} \mapsto \mathcal{S}^c$ on the category \mathbf{SCR} ; by \mathcal{R}^c we shall mean the pro-simplicial ring (\mathcal{R}_α^c) , obtained by applying c level-wise.

We will quite frequently want to relate the functor represented by \mathcal{R} to the functor represented by $\pi_0 \mathcal{R}$. We shall need this comparison both when $\mathcal{R} \in \mathbf{Art}_k$ and when $\mathcal{R} \in \mathbf{pro}\text{-}\mathbf{Art}_k$; in the latter case $\pi_0 \mathcal{R}$ is a pro-finite set. Even if $\mathcal{R} \in \mathbf{Art}_k$ is cofibrant, the ordinary ring $\pi_0 \mathcal{R}$, regarded as a (constant) simplicial ring is not, but using the chosen cofibrant approximation functor we have a comparison zig-zag $(\pi_0 \mathcal{R})^c \leftarrow \mathcal{R}^c \rightarrow \mathcal{R}$, inducing a zig-zag of functors

$$\mathrm{Hom}_{\mathbf{Art}_k}((\pi_0 \mathcal{R})^c, -) \rightarrow \mathrm{Hom}_{\mathbf{Art}_k}(\mathcal{R}^c, -) \xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Art}_k}(\mathcal{R}, -).$$

Thus, for example, if \mathcal{F} is representable with representing object \mathcal{R} , we get

$$(7.6) \quad \mathrm{Hom}_{\mathbf{Art}_k}((\pi_0 \mathcal{R})^c, -) \dashrightarrow \mathcal{F}.$$

If \mathcal{R} is a pro-object of \mathbf{Art}_k the same construction applies levelwise.

Finally, we will often encounter situations of the following type. Consider a sequence of maps in $\mathbf{pro}\text{-}\mathbf{Art}_k$

$$(7.7) \quad \mathcal{R}_1 \xleftarrow[g]{\sim} \mathcal{R}_2 \xrightarrow{h} \mathcal{R}_3.$$

where \sim in the first map means that it induces an objectwise weak equivalence $\mathrm{Hom}(\mathcal{R}_1, -) \rightarrow \mathrm{Hom}(\mathcal{R}_2, -)$ of functors $\mathbf{Art}_k \rightarrow \mathbf{sSets}$. If \mathcal{R}_3 is nice, we can “invert” the first arrow, in a homotopical sense, to get a morphism $\mathcal{R}_1 \rightarrow \mathcal{R}_3$ in $\mathbf{pro}\mathbf{Art}_k$, in the sense that we produce

$$f : \mathcal{R}_1 \rightarrow \mathcal{R}_3$$

such that $f \circ g$, although not literally equal to h , induces the same map on $\pi_0 \mathrm{Hom}(-, \mathcal{A})$. Thus, for example, $f \circ g$ and h induce the same map on tangent complexes, and the same map in the pro-homotopy category (see discussion of §3.5).

To produce $\mathcal{R}_1 \rightarrow \mathcal{R}_3$, consider the map of functors

$$\mathrm{Hom}(\mathcal{R}_3, -) \rightarrow \mathrm{Hom}(\mathcal{R}_2, -) \xleftarrow{\sim} \mathrm{Hom}(\mathcal{R}_1, -)$$

and as in (7.3) replace $\mathrm{Hom}(\mathcal{R}_3, -)$ by a naturally weakly equivalent functor $\mathcal{F}_3 \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R}_3, -)$ together with a map $\mathcal{F}_3 \rightarrow \mathrm{Hom}(\mathcal{R}_1, -)$. Now \mathcal{R}_3 is still a representing ring for \mathcal{F}_3 : there is a natural weak equivalence $\mathrm{hocolim} \mathrm{Hom}(\mathcal{R}_{3,\alpha}, -) \rightarrow \mathcal{F}_3$ by Lemma 2.21. Then apply Lemma 2.25 to produce a map $\mathcal{R}_1 \rightarrow \mathcal{R}_3$; the resulting diagrams of functors all commute objectwise in the homotopy category.

7.2. Tangent complex. For a (possibly pro-) simplicial ring \mathcal{R} augmented over k , we define the tangent complex $\mathfrak{t}\mathcal{R}$ to be the tangent complex of the associated functor $\mathrm{Hom}(\mathcal{R}, -)$, where the maps are taken in SCR/k , and we use $\mathfrak{t}^i\mathcal{R}$ to abbreviate $\pi_{-i}\mathfrak{t}\mathcal{R}$. (As before we shall apply this definition only when \mathcal{R} is cofibrant.) Recall that $\mathfrak{t}\mathcal{R}$ is a chain complex with degree-decreasing differentials, with homology supported entirely in degrees ≤ 0 ; therefore $\mathfrak{t}^i\mathcal{R}$ is supported in $i \geq 0$.

To be explicit, if $\mathcal{R} = (\mathcal{R}_\alpha)$, where \mathcal{R}_α are all cofibrant, we have

$$(7.8) \quad \mathfrak{t}^i\mathcal{R} = \varinjlim_{\alpha} \pi_{j-i} \mathrm{Hom}_{\mathrm{Art}_k}(\mathcal{R}_\alpha, k \oplus k[j]), \text{ any fixed } j \geq i.$$

Let $D_A^i(B, M)$ denote the André-Quillen cohomology functors of the A -algebra B with coefficients in the B -module M (see discussion before Example 4.27, or [27] – in particular D^0 is “usual” derivations). In the case of a simplicial ring \mathcal{R} augmented over k , we have by Example 4.27 an identification

$$\mathfrak{t}^i\mathcal{R} = D_{\mathbb{Z}}^i(\mathcal{R}, k)$$

and in the “pro” case there is a corresponding assertion with \varinjlim .

If $\mathcal{R} \rightarrow \mathcal{S}$ is a morphism of simplicial commutative rings (we won’t need the “pro” case) we get a long exact sequence

$$(7.9) \quad D_{\mathcal{R}}^i(\mathcal{S}, k) \rightarrow \mathfrak{t}^i\mathcal{S} \rightarrow \mathfrak{t}^i\mathcal{R} \xrightarrow{[1]}$$

(this follows from the long exact sequence for André-Quillen cohomology, [27, Theorem 5.1]).

Let $\mathcal{R} \in \mathrm{Art}_k$. Let $\pi_0\mathcal{R}/p^n$ be the quotient of $\pi_0\mathcal{R}$ by the ideal generated by p^n in the usual (non-derived) sense of quotient of a ring by an ideal; we shall consider the result as a discrete simplicial ring. Then we have natural isomorphisms

$$(7.10) \quad \mathfrak{t}^0\mathcal{R} \cong \mathfrak{t}^0(\pi_0\mathcal{R}) \cong \mathfrak{t}^0(\pi_0\mathcal{R}/p^n) \quad (n \geq 1),$$

because the map $\mathcal{R} \rightarrow \pi_0\mathcal{R} \rightarrow \pi_0\mathcal{R}/p^n$ all induce isomorphisms when we consider homomorphisms into $k[x]/x^2$, considered as a discrete simplicial ring. Indeed since any homomorphism $\pi_0\mathcal{R} \rightarrow k[x]/x^2$ is trivial on $(p, \mathfrak{m}_{\pi_0\mathcal{R}}^2)$, with $\mathfrak{m}_{\pi_0\mathcal{R}}$ the kernel of $\pi_0\mathcal{R} \rightarrow k$, we could have equally well replaced $\pi_0\mathcal{R}/p^n$ by $\pi_0\mathcal{R}/\mathfrak{b}$ so long as $\mathfrak{b} \subset (p, \mathfrak{m}_{\pi_0\mathcal{R}}^2)$.

Similarly, the map $\mathcal{R} \rightarrow \pi_0\mathcal{R}$ always induces an injection

$$(7.11) \quad \mathfrak{t}^1(\pi_0\mathcal{R}) \hookrightarrow \mathfrak{t}^1\mathcal{R}.$$

Informally (7.11) arises from the fact that $\pi_0\mathcal{R}$ can be presented as a \mathcal{R} -algebra by freely adding relations in degree 2 and higher to kill all higher homotopy. More formally, this follows from Proposition 4.3 part (ii): writing $\mathcal{R}' = \pi_0\mathcal{R}$, considered as a discrete simplicial ring, we have an exact sequence

$$\cdots \rightarrow \pi_1(\mathcal{R}') \rightarrow \pi_1(\mathcal{R}', \mathcal{R}) \rightarrow \pi_0\mathcal{R} \rightarrow \pi_0\mathcal{R}' \rightarrow \pi_0(\mathcal{R}', \mathcal{R}) \rightarrow 0.$$

and thus $\pi_j(\mathcal{R}', \mathcal{R}) = 0$ for $j = 0, 1$; now the quoted Proposition shows that $\pi_{-1}t(\mathcal{R}', \mathcal{R}) = 0$, and then (4.2) gives the desired result.

The assertion of (7.10) and (7.11) continue to hold for a pro-simplicial commutative ring (\mathcal{R}_α) , where we define:

$$(7.12) \quad \pi_0\mathcal{R} = \text{the pro-ring } (\alpha \mapsto \pi_0\mathcal{R}_\alpha).$$

7.3. The Galois representation. We will be interested in analyzing the derived deformation ring of a representation

$$\rho : \Gamma_S \rightarrow G(k),$$

Eventually ρ will be a Galois representation associated to a cohomology class, i.e. it will be the $\bar{\sigma}$ from Conjecture 6.1, and it will moreover be required to satisfy some “big image” and “nice at p ” conditions that are formulated in §10. However, for the moment, we do not impose any restrictions on ρ .

We write $\text{Ad } \rho$ for the adjoint representation of Γ_S on the Lie algebra \mathfrak{g} of G over k , i.e. the composition of ρ with the adjoint representation of \mathbf{G} :

$$\text{Ad } \rho : \Gamma_S \rightarrow \text{GL}(\mathfrak{g}).$$

We will also be concerned with the k -linear dual of this representation, which acts on the k -linear dual \mathfrak{g}^* to \mathfrak{g} ; we denote it Ad^* :

$$\text{Ad}^* \rho : \Gamma_S \rightarrow \text{GL}(\mathfrak{g}^*).$$

Our basic object of interest will be the deformation functor corresponding to deforming $\rho : \Gamma_S \rightarrow G(k)$. In the notation of Definition 5.4 (iii) this is the functor $\mathcal{F}_{X, \rho} : \text{Art}_k \rightarrow s\text{Sets}$ where⁵

$$X = \text{étale homotopy type of } \text{Spec} \mathbb{Z}[\frac{1}{S}] \in \text{pro}(s\text{Sets}).$$

We will denote this deformation functor by $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}], \rho}$. Explicitly it's given as

$$(7.13) \quad \mathcal{F}_{\mathbb{Z}[\frac{1}{S}], \rho} : A \mapsto \text{fiber of } \varinjlim_\alpha \text{Hom}_{s\text{Sets}}(X_\alpha, BG(A)) \text{ above } \rho,$$

where we wrote X (= étale homotopy type of $\mathbb{Z}[\frac{1}{S}]$) as the pro-simplicial set (X_α) , and where $BG(A)$ for a simplicial ring A has been described in Definition 5.1. Informally $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}], \rho}$ sends an Artinian simplicial ring A to a simplicial set of “conjugacy classes of deformations of $\rho : \Gamma_S \rightarrow G(k)$ to A .”

⁵in fact, for our purposes, we could also replace X by the pro-simplicial set X' which is the inverse system $(BG_\alpha)_\alpha$, where we write $\pi_1\mathbb{Z}[\frac{1}{S}] = \varprojlim_\alpha G_\alpha$ as an inverse limit of finite groups. There is a natural map $X \rightarrow X'$ which need not be an isomorphism; nonetheless, they give the same deformation functor, because the étale cohomology groups coincide with p -torsion coincides by the equality of étale and Galois cohomology with such coefficients [25, Proposition 2.9].

There is also a framed version where one does not quotient by conjugacy:

$$\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}^{\square} : \text{Art}_k \rightarrow s\text{Sets}$$

which has been described in Definition 5.4 (i), but informally sends an Artinian simplicial ring A to a simplicial set of “lifts of $\rho : \Gamma_S \rightarrow G(k)$ to A ,” i.e. we don’t quotient by conjugacy.

Just as for rings (§7.2) we use the notation $\mathfrak{t}^j \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ for the $(-j)$ th homotopy group $\pi_{-j} \mathfrak{t} \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ of the tangent complex of $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$. Recall that this is $\pi_0 \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}(k \oplus k[j])$ by definition and that we have an identification (Lemma 5.10, specialized to the case of trivial center)

$$\mathfrak{t}^j \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho} = \pi_{-j} \mathfrak{t} \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho} \cong H^{j+1}(\mathbb{Z}[\frac{1}{S}], \text{Ad} \rho), j \geq -1$$

between the the homotopy groups of the tangent complex of $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ and the adjoint cohomology. Recall our convention above that $H^*(\mathbb{Z}[\frac{1}{S}], -)$ denotes the étale cohomology.

Recall (§4.3) that the groups \mathfrak{t}^j are *a priori* the homotopy groups of a certain spectrum, but, by the discussion of §4.3, we can canonically (but not very explicitly) consider them as the homology of a chain complex of k -vector spaces, which we shall refer to as the tangent complex.

We say that ρ is *Schur* if its centralizer coincides with the center of G . (For example, for $G = \text{GL}_n$, this would be implied by absolute irreducibility of ρ ; in our case, we are supposing G is adjoint, and so we are requiring that the centralizer of ρ is trivial.) In that case \mathfrak{t}^j is vanishing for negative j and so Lurie’s derived Schlessinger criterion (Theorem 4.33) implies that $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ is represented by a pro-object in Art_k ; we have reviewed what that means in §7.1.

Finally, an important remark is that $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ recovers the usual deformation functor of Mazur upon passage to π_0 :

Lemma 7.1. *Let discArt_k be the category of usual (i.e. discrete) Artin local rings A equipped with an identification of their residue field to k . Suppose also that ρ is Schur, i.e. has trivial centralizer. Then the functor*

$$(7.14) \quad \pi_0 \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho} : \text{discArt}_k \longrightarrow \text{Sets}$$

obtained by sending a usual Artin ring A to $\pi_0 \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}(A)$ is naturally isomorphic to the usual, underived deformation functor; that is to say

$$(7.15) \quad A \rightarrow \text{lifts of } \rho \text{ to } G(A)/\text{conjugation by } \ker(G(A) \rightarrow G(k)),$$

This result implies, in particular, that π_0 of a representing ring for $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ recovers Mazur’s deformation ring. There is a very similar result for framed deformation rings, where one does not need to assume that ρ has trivial centralizer.

Proof. Let X be the étale homotopy type of $\mathbb{Z}[\frac{1}{S}]$. Then the functor $\mathcal{F}_{X,G}$ of Definition 5.4 sends A to the mapping space $\mathrm{Hom}_{s\mathrm{Sets}}(X, BG(A))$ (see remark after Definition 5.1).

The components of the mapping space $\mathrm{Hom}_{s\mathrm{Sets}}(X, BG(A))$ are identified with the set of $G(A)$ -torsors over X ; these are, in turn, in bijection with conjugacy classes of maps

$$(7.16) \quad \tilde{\rho} : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(A).$$

Thus the functor $\mathcal{F}_{X,\rho}$ of Definition 5.4, sends $A \in \mathrm{discArt}_k$ to the subset of (7.16) comprising those $\tilde{\rho}$ whose reduction to $G(k)$ is conjugate to ρ ; that is identified with the right-hand side of (7.15). \square

7.4. Notation for deformation rings and deformation functors. We will use the phrase “deformation ring” to mean a ring representing a deformation functor, if the functor is representable. For example, the deformation ring of $\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$, is, by definition, the pro-object of Art_k representing the functor $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$, if that functor is indeed representable.

We need to study not just the deformation ring of $\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$, but also the deformation ring of ρ pulled back to various other groups, e.g. the fundamental group of \mathbb{Q}_q . This gives rise to a small zoo of deformation functors and rings. We briefly summarize the notation for them, although we will also define them again as we use them. In particular, we shall follow the

Notational convention: We will always be deforming the same representation ρ , but pulled back to various other groups besides $\pi_1 \mathbb{Z}[\frac{1}{S}]$. Therefore, we will omit completely ρ from the notation for deformation rings, and rather keep track of what ρ has been pulled back to. Thus, we will abridge $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho}$ simply to $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}$, or even to \mathcal{F}_S , and similarly for the representing ring; thus

$$\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} = \mathcal{F}_S \text{ for short, } \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} = \mathcal{R}_S \text{ for short}$$

will denote the deformation functor and deformation ring for ρ considered as a representation of $\pi_1 \mathbb{Z}[\frac{1}{S}]$. As a warning (see discussion below) we note that the meaning of these notations is changed after §10: they will always refer to the functor and ring with *crystalline* conditions imposed.

Now we can pull back ρ under $\pi_1 \mathbb{Q}_q \rightarrow \pi_1 \mathbb{Z}[\frac{1}{S}]$ or, if $q \notin S$, under $\pi_1 \mathbb{Z}_q \rightarrow \pi_1 \mathbb{Z}[\frac{1}{S}]$; we denote the resulting representations as $\rho_{\mathbb{Q}_q}$ and $\rho_{\mathbb{Z}_q}$, and the resulting deformation functor (for $\pi_1 \mathbb{Q}_q$ or $\pi_1 \mathbb{Z}_q$, respectively) by $\mathcal{F}_{\mathbb{Q}_q}$ or $\mathcal{F}_{\mathbb{Z}_q}$. These are defined just as in (7.13), but replacing the role of $\mathbb{Z}[\frac{1}{S}]$ by \mathbb{Q}_q or \mathbb{Z}_q . These functors are rarely representable. The corresponding framed functors are representable, leading to a deformation functor and representing ring

$$\mathcal{F}_{\mathbb{Q}_q}^{\square}, \mathcal{R}_{\mathbb{Q}_q}^{\square}$$

and similarly for \mathbb{Z}_q .

Later on we will consider certain sets Q_n of auxiliary primes, disjoint from S , and will want to consider ρ as a representation of $\pi_1 \mathbb{Z}[\frac{1}{SQ_n}]$. Again we can define deformation functors as in (7.13), replacing the role of $\mathbb{Z}[\frac{1}{S}]$ by $\mathbb{Z}[\frac{1}{SQ_n}]$. These deformation functors and rings will be denoted by

$$\mathcal{F}_{S \amalg Q_n} = \mathcal{F}_n \text{ for short, } \mathcal{R}_{S \amalg Q_n} = \mathcal{R}_n \text{ for short,}$$

where we will only use the abridged forms when Q_n is understood.

Finally, from §10 onwards, we will *only* consider the deformation functors with crystalline conditions imposed at p (the precise meaning of this is spelled out in §9). To avoid notational overload we will not explicitly include this in the notation. Therefore, from §10 onwards, the notations $\mathcal{F}_S, \mathcal{R}_S, \mathcal{F}_{S \amalg Q_n}$ etc. always denote the versions of these functors and rings with crystalline conditions imposed at p .

7.4.1. Representations with target T : the rings \mathcal{S} . Finally, later on, we will study various deformation rings with targets not in the algebraic group G but just in its torus T .

In particular, our representation functors with target T will *never* be representable, because they always have automorphisms; but the framed versions will be. We will use the notation \mathcal{S} for deformation rings representing framed deformation functors with target T . This notation will be specified precisely when we use it, but typical examples will be the following:

For certain “Taylor–Wiles” primes $q \notin S$, we will choose a representation $\rho_{\mathbb{Z}_q}^T : \pi_1 \mathbb{Z}_q \rightarrow T(k)$ which is conjugate inside $G(k)$, to $\rho_{\mathbb{Q}_q}$ (see (6.7)); let $\rho_{\mathbb{Q}_q}^T : \pi_1 \mathbb{Q}_q \rightarrow T(k)$ be its pullback to \mathbb{Q}_q . We write

$$\mathcal{S}_q = \text{framed deformation ring of } \rho_{\mathbb{Z}_q}^T \text{ pulled back to } \pi_1 \mathbb{Q}_q,$$

$$\mathcal{S}_q^{\text{ur}} = \text{framed deformation ring of } \rho_{\mathbb{Z}_q}^T,$$

the superscript ur is for “unramified.”

For example, just to be completely explicit \mathcal{S}_q represents the functor $\mathcal{F}_{X, \rho_{\mathbb{Z}_q}^T}^\square : \text{Art}_k \rightarrow s\text{Sets}$ from Definition 5.4, where X is now the étale homotopy type of \mathbb{Q}_q , and we replace BG by BT in that Definition.

7.4.2. Mnemonics. In summary we have the following general mnemonic for our notations:

$$\mathcal{F}_? = \text{deformation functor for } \rho, \text{ with target } G, \text{ at level } ?,$$

(if $? = n$, this is a shorthand for level $S \amalg Q_n$ for a set of auxiliary primes Q_n);

$$\mathcal{R}_? = \text{representing ring for } \mathcal{F}_?,$$

$$\mathcal{R}_?^\square = \text{representing ring for framed version of } \mathcal{F}_?,$$

$$\mathcal{S} = \text{a representing ring for a framed deformation problem with target } T$$

superscript ur = “unramified”, e.g. deformations of $\pi_1 \mathbb{Z}_q$ as opposed to $\pi_1 \mathbb{Q}_q$

We will also use roman letters for the corresponding “usual” deformation rings. For example, just as \mathcal{R}_S denotes the deformation functor for ρ with target G at level $\mathbb{Z}[\frac{1}{S}]$, $R_S = \pi_0 \mathcal{R}_S$ denotes the corresponding (Mazur) deformation ring.

7.5. Pro-rings and complete local rings. In the usual (non-simplicial) setting of deformation theory, it does not matter much whether one allows functors from Artin rings to be represented by a pro-object in Artin rings, or whether one allows the “representing” object to be the complete ring slightly outside the category of Artin rings. The latter is the traditional choice.

In our simplicial setting, we have chosen to consistently work with pro-objects. It is certainly possible to work with complete objects in this setting too, as does Lurie [22]; however, we will always stick to the pro-object point of view when we deal with simplicial rings and derived deformation problems.

It will be convenient for us to review certain aspects of the passage between the “complete” and “pro” point of view in the case of usual (non-simplicial) deformation theory:

Lemma 7.2. *Let $R = (R_\alpha)$ be a sequentially indexed pro-object of the category of (usual) Artin local rings augmented over k . Suppose that $\varinjlim \text{Hom}(R_\alpha, k[\epsilon]/\epsilon^2)$ is finite-dimensional (i.e., ${}^t R$ is finite-dimensional). Then $R := \varprojlim R_\alpha$ is a complete local Noetherian ring.*

We will refer to R as the “associated complete local ring” to the pro-object R_α .

Proof. We can suppose all the transition maps on R_α are actually surjective, otherwise we just replace R_α by the intersection of all the images of the maps from $\beta > \alpha$.

Write $\mathfrak{m}_R = \ker(R \rightarrow k) = \varprojlim \mathfrak{m}_\alpha$, where \mathfrak{m}_α is the kernel of $R_\alpha \rightarrow k$. Then $R/\mathfrak{m}_R \cong k$, and moreover \mathfrak{m}_R is clearly the unique maximal ideal, for any element not in it is invertible.

Write J_α for the kernel of the natural map from R to R_α . Thus $R = \varprojlim R/J_\alpha$. Because R_α is Artin local, $\mathfrak{m}_\alpha^{k_\alpha} = 0$ for some k_α . Therefore, $\mathfrak{m}_R^{k_\alpha} \subset J_\alpha$. Therefore, R is complete with respect to the topology defined by powers of the maximal ideal. (In fact, this topology is readily verified to coincide with the profinite topology.)

By assumption, $\varinjlim (\mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2)^\vee$ is finite dimensional, say, of dimension s . Since the transition maps are injective, the k -dimension of $\mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2 \leq s$ for all α .

We claim that in fact

$$(7.17) \quad \mathfrak{m}_R^t = \varprojlim \mathfrak{m}_\alpha^t.$$

Assuming this we get (everything is finite!) $\mathfrak{m}_R/\mathfrak{m}_R^2 = \varprojlim \mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2$, which implies that R is a quotient of the ring $W(k)[[X_1, \dots, X_s]]$ and so Noetherian.

To verify (7.17), first note that multiplication induces a surjection of k -vector spaces $(\mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2)^{\otimes t} \rightarrow \mathfrak{m}_\alpha^t/\mathfrak{m}_\alpha^{t+1}$ and so the map

$$\mathfrak{m}_R^t \rightarrow \mathfrak{m}_\alpha^t/\mathfrak{m}_\alpha^{t+1}$$

is surjective. Suppose $Y_\alpha \in \mathfrak{m}_\alpha^t$ form a compatible system; there is $X_0 \in \mathfrak{m}_R^t$ such that $X_0 \equiv Y_\alpha$ modulo $\mathfrak{m}_\alpha^{t+1}$. We can then find $X_1 \in \mathfrak{m}_R^{t+1}$ such that $X_1 \equiv Y_\alpha - X_0$ modulo $\mathfrak{m}_\alpha^{t+2}$. Proceeding in this way, the series $X_0 + X_1 + \dots$ converges to an element of R , and its limit is equal to Y_α modulo all powers of \mathfrak{m}_α , as desired. \square

In that setting the following lemma will be useful, to compare the tangent complex of a pro-object with a tangent complex of the corresponding complete ring:

Lemma 7.3. *Suppose that R is a (usual) complete local Noetherian ring with maximal ideal \mathfrak{m} , equipped with an isomorphism $R/\mathfrak{m} \rightarrow k$. Then $\mathfrak{t}^*R \cong \varinjlim \mathfrak{t}^*(R/\mathfrak{m}^N)$.*

In particular, with the notations of the Lemma, the functor $\mathrm{Art}_k \rightarrow \mathbf{sSets}$ given by

$$A \mapsto \mathrm{colim}_{N \rightarrow \infty} \mathrm{Art}_k(R/\mathfrak{m}^N, A)$$

has i th tangent cohomology given by $\mathfrak{t}^i R = D_{\mathbb{Z}}^i(R, k) = D_{W(k)}^i(R, k)$. (For the last equality we look at the long exact sequence in André–Quillen cohomology for $\mathbb{Z} \rightarrow W(k) \rightarrow k$: that shows that $D_{\mathbb{Z}}^i(W(k), k) = 0$. This allows us to compare $D_{\mathbb{Z}}^i(R, k)$ and $D_{W(k)}^i(R, k)$, as desired.)

We will apply this later in the case where $R = W(k)[[X_1, \dots, X_s]]/(Y_1, \dots, Y_t)$ and the Y_i are a regular sequence inside the maximal ideal (p, X_1, \dots, X_s) of $W(k)[[X_1, \dots, X_s]]$. Then, if the reductions \overline{Y}_i to $k[[X_1, \dots, X_s]]$ have only quadratic and higher terms, standard computations of André–Quillen cohomology imply that

$$(7.18) \quad \dim_k \mathfrak{t}^i R = \begin{cases} s, & i = 0 \\ t, & i = 1, \\ 0, & \text{else} \end{cases}$$

Proof. (of Lemma 7.3): It is enough to show that the induced map of André–Quillen cohomology (denoted D^i), with coefficients in k ,

$$\varinjlim D_{\mathbb{Z}}^i(R_N, k) \rightarrow D_{\mathbb{Z}}^i(R, k)$$

is an isomorphism. From the long exact sequence [27, Theorem 5.1] associated to $\mathbb{Z} \rightarrow R \rightarrow R_N$ it is enough to see that, for all i ,

$$\varinjlim D_R^i(R_N, k) = 0$$

which is a consequence of [27, Theorem 6.15] (unfortunately there is no proof given). \square

We will want a reasonable way to say that a pro-simplicial ring is discrete. We do not need to discuss any homotopy theory of pro-simplicial rings: for us their importance is solely in the functors they represent, and the following definition is adequate:

Definition 7.4. *We say that a pro-object $\mathcal{R} = \{\mathcal{R}_\alpha\}_\alpha$ of Art_k is homotopy discrete if the map $p : \mathcal{R} \rightarrow \pi_0 \mathcal{R}$ induces an equivalence on represented functors (after applying level-wise cofibrant replacement), i.e. if the map*

$$\mathrm{colim}_\alpha \mathrm{Hom}_{\mathrm{Art}_k}(\pi_0 \mathcal{R}_\alpha^c, A) \rightarrow \mathrm{colim}_\alpha \mathrm{Hom}_{\mathrm{Art}_k}(\mathcal{R}_\alpha^c, A)$$

is a weak equivalence for all $A \in \text{Art}_k$.

The following Lemma will show that, in the arithmetic contexts where we apply it, the coincidence of “derived” and “usual” deformation rings is equivalent to the usual deformation ring being a complete intersection of the expected size:

Lemma 7.5. *Suppose $\mathcal{R} = \{\mathcal{R}_\alpha\}_\alpha$ is an object of pro-Art_k such that $b_i := \dim(\mathfrak{t}^i \mathcal{R})$ is finite for all i , and zero if $i \notin \{0, 1\}$. Denote by $\pi_0 \mathcal{R} = (\alpha \mapsto \pi_0 \mathcal{R}_\alpha)$ the associated pro-ring. In this case, the following are equivalent:*

- (i) \mathcal{R} is homotopy discrete, in the sense of Definition 7.4.
- (ii) The complete local ring $\varprojlim_\alpha \pi_0 \mathcal{R}_\alpha$ associated to $\pi_0 \mathcal{R}$ is isomorphic to

$$W(k)[[X_1, \dots, X_{b_0}]]/(Y_1, \dots, Y_{b_1})$$

for a regular sequence Y_i of elements all belonging to (p, \mathfrak{m}^2) (where \mathfrak{m} is the maximal ideal of $W(k)[[X_*]]$).

Proof. For (ii) implies (i): the map $\mathcal{R} \rightarrow \pi_0 \mathcal{R}$ always induces an isomorphism on \mathfrak{t}^0 , and an injection on \mathfrak{t}^1 (see (7.11)); it is an isomorphism on \mathfrak{t}^1 then by dimension-counting (see Lemma 7.3 for why we can compute the tangent complex of the pro-ring $\pi_0 \mathcal{R}$ in terms of the tangent complex of the associated complete local ring.)

For (i) implies (ii): Let $R = \varprojlim_\alpha \pi_0 \mathcal{R}_\alpha$. Under the quoted finiteness assumption, R is a complete Noetherian local ring (Lemma 7.2). By assumption (i), together with Lemma 7.3, we have $\dim D_{W(k)}^i(R, k) = b_i$. Thus there is a surjection of complete local Noetherian rings

$$W(k)[[X_1, \dots, X_{b_0}]] \twoheadrightarrow R$$

compatible with the augmentations to k . Any element Y in the kernel has the property that $\bar{Y} \in k[[X_i]]$ doesn’t involve any linear terms, otherwise $\dim \mathfrak{t}^0 R$ would be too small. Thus $Y \in (p, \mathfrak{m}^2)$. What we must show is that the kernel is generated by a regular sequence; once this is so, the length of the regular sequence must be $\dim D_{W(k)}^1(R, k) = b_1$. For this, see [18, Theorem 8.5] and its proof in particular. \square

We conclude with a minor result about projective limits of Tor-groups (which is mainly cosmetic).

Lemma 7.6. *Let S be a complete local ring, and I_n a sequence of ideals which form a basis for the topology of S . Let M, N be finitely generated complete S -modules. Set $M_k = M/I_k$, $N_k = N/I_k$, $S_k = S/I_k$ and suppose these are all finite (as sets). Then the natural map*

$$\text{Tor}_S(M, N) \rightarrow \varprojlim_k \text{Tor}_{S_k}(M_k, N_k)$$

is an isomorphism.

Proof. First of all, the map

$$(7.19) \quad \text{Tor}_S(M, N) \rightarrow \varprojlim_k \text{Tor}_S(M, N_k)$$

is an isomorphism. To see this, choose a resolution F_\bullet of M by finite free S -modules; tensoring with N , our result follows from the fact that $N = \varprojlim N_k$ and homology commutes with inverse limits for complexes of finite abelian groups.

We are now reduced to verifying that the natural map

$$\mathrm{Tor}_S(M, N) \rightarrow \varprojlim_{n \geq k} \mathrm{Tor}_{S_n}(M_n, N)$$

is an isomorphism when N is an S_k -module.

By change of rings [32, Tag 068F] (in the derived sense, we have $M \otimes_S N = M \otimes_S S_n \otimes_{S_n} N$) we get there is a spectral sequence computing $\mathrm{Tor}_S^r(M, N)$:

$$(7.20) \quad \bigoplus_{p+q=r} \mathrm{Tor}_{S_n}^p(\mathrm{Tor}_{S_n}^q(M, S_n), N)$$

All the terms in this spectral sequence are finite abelian groups. Also, this spectral sequence is “compatible with change of n ,” i.e. there is a map of spectral sequences from the spectral sequence at level N to the spectral sequence at level $n < N$. Thus we can take the inverse limit and obtain a spectral sequence with first page

$$(7.21) \quad \varprojlim_n \bigoplus_{p+q=r} \mathrm{Tor}_{S_n}^p(\mathrm{Tor}_{S_n}^q(M, S_n), N)$$

which still computes $\mathrm{Tor}_S(M, N)$.

Because $\varprojlim_n \mathrm{Tor}_q^S(M, S_n) = 0$ in positive degree q , by (7.19) with $N = S$, the natural maps

$$\mathrm{Tor}_q^S(M, S_n) \rightarrow \mathrm{Tor}_q^S(M, S_n)$$

must be zero for fixed n and big enough N .

That means that, in the (7.21) all the terms with $q > 0$ go away. Thus the natural edge map

$$\mathrm{Tor}_S(M, N) \xrightarrow{\sim} \varprojlim_n \mathrm{Tor}_{S_n}^0(M \otimes_S S_n, N)$$

which is what we wanted. \square

8. ALLOWING RAMIFICATION AT EXTRA PRIMES

In this section we examine the behavior of the derived deformation ring when adding a prime to the set of ramification. The statement is the obvious one – it is a translation, at the level of deformation rings, of the statement “a representation of $\pi_1(\mathbb{Z}[\frac{1}{Sq}])$ unramified at q is actually a representation of $\pi_1(\mathbb{Z}[\frac{1}{S}])$.”

We need this only for certain primes q , Taylor–Wiles primes – see §6.7 – which present a particularly simple situation as far as local deformation theory goes.

8.1. Notation and the main result. We briefly recall the relevant notation from the previous section, especially §7.3 and §7.4: we’re deforming a fixed representation $\rho : \Gamma_S \rightarrow G(k)$ from $\Gamma_S = \pi_1^{\mathrm{et}}(\mathbb{Z}[1/S])$ into the k -points of the reductive group G . We denote by $\mathcal{F}_S = \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}$ the corresponding deformation functor (as discussed in §7.4, because ρ is always fixed, we omit it from the notation). Let

now q be a Taylor–Wiles prime, as in §6.7. Denote by $\rho_{\mathbb{Z}_q}$ the pullback of ρ under (the map of étale fundamental groups associated with) $\mathrm{Spec}\mathbb{Z}_q \rightarrow \mathrm{Spec}\mathbb{Z}[\frac{1}{S}]$. By assumption, $\rho_{\mathbb{Z}_q}$ is conjugate, in $G(k)$, to a representation of the étale fundamental group of \mathbb{Z}_q into $T(k)$. As part of the data associated to a Taylor–Wiles prime, we have fixed such a representation $\rho_{\mathbb{Z}_q}^T$, *together with* an isomorphism of

$$(8.1) \quad \text{inclusion}_T^G \circ \rho_{\mathbb{Z}_q}^T \cong \rho_{\mathbb{Z}_q},$$

and our constructions that follow will depend on this specific isomorphism.

We define similarly $\rho_{\mathbb{Q}_q}, \rho_{\mathbb{Q}_q}^T$ by pullback via $\pi_1\mathbb{Q}_q \rightarrow \pi_1\mathbb{Z}_q$.

Then make the following definitions:

- $\mathcal{F}_{\mathbb{Z}_q}, \mathcal{F}_{\mathbb{Q}_q}$ are the deformation functors for $\rho_{\mathbb{Z}_q}$ and $\rho_{\mathbb{Q}_q}$, considered as representations into $G(k)$;
- $\mathcal{F}_{\mathbb{Z}_q}^T$ or $\mathcal{F}_{\mathbb{Q}_q}^T$ are deformation functors for $\rho_{\mathbb{Z}_q}^T$ or $\rho_{\mathbb{Q}_q}^T$ *valued in* T (and recall the convention about T -valued deformation functors from §7.4).
- The choice of isomorphism of (8.1) induces a map $\mathcal{F}_{\mathbb{Z}_q}^T \rightarrow \mathcal{F}_{\mathbb{Z}_q}$, similarly for \mathbb{Q}_q .
- $\mathcal{F}_{\mathbb{Z}_q}^{T,\square}$ or $\mathcal{F}_{\mathbb{Q}_q}^{T,\square}$ are framed deformation functors for $\rho_{\mathbb{Z}_q}^T$ or $\rho_{\mathbb{Q}_q}^T$ *valued in* T .
- (Ignore this one until you need to read it:) Let S_q° be the usual (underived) framed deformation ring for the trivial representation $I_q \rightarrow T$. See (6.2) for the definition of $I_q \cong (\mathbb{Z}/q)^*$.

Our main results can be summarized in the following diagram, where all squares are object-wise homotopy pullback squares (see (7.4) for definition), the s -maps are sections of the natural maps that they are adjacent to, and the symbol \sim means objectwise weak equivalence. The maps are the “natural ones”, except for s, j which are discussed in §8.4 and §8.5 respectively; the notation $(S_q^\circ)^c$ means a cofibrant replacement for S_q° , where the latter is considered as a discrete simplicial commutative ring. The square (c) is a homotopy pullback considered both ways, i.e. both with the “left to right” horizontal arrows and with the “right to left” horizontal actions.

$$(8.2) \quad \begin{array}{ccccccc} \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & \mathcal{F}_{\mathbb{Z}_q} & \xleftarrow{\sim} & \mathcal{F}_{\mathbb{Z}_q}^T & \xleftarrow{s_{\mathbb{Z}_q}} & \mathcal{F}_{\mathbb{Z}_q}^{T,\square} & \longrightarrow & * \\ \downarrow & (a) & \downarrow & (b) & \downarrow & (c) & \downarrow & (d) & \downarrow \\ \mathcal{F}_{\mathbb{Z}[\frac{1}{S_q}]} & \longrightarrow & \mathcal{F}_{\mathbb{Q}_q} & \xleftarrow{\sim} & \mathcal{F}_{\mathbb{Q}_q}^T & \xleftarrow{s_{\mathbb{Q}_q}} & \mathcal{F}_{\mathbb{Q}_q}^{T,\square} & \xrightarrow{j} & \mathrm{Hom}_{\mathrm{Art}_k}((S_q^\circ)^c, -) \end{array}$$

Remark 8.1. *Strictly speaking, this diagram is not correct: it exists only after replacing some of the functors by naturally weakly equivalent ones; the diagram as above exists literally only in the homotopy category, i.e. at the cost of replacing*

functors by their π_0 . We have allowed ourselves to be imprecise about this since they do not affect the key deduction (8.3) that follows.

We briefly describe the sources of this: Firstly, to produce the splittings of square (c), we must replace the functors occurring there by naturally weakly equivalent functors, as mentioned in the discussion around (8.11). Secondly, in square (d), we must insert some homotopy commutative squares with horizontal weak equivalences; these arise from passing from functors to representing rings, as in Lemma 2.25. Also, in square (d), the map j exists only as a zig-zag, for reasons similar to those mentioned near (7.6).

Remark 8.2. The first square (a) is a homotopy pullback square for any prime q . It is only in the analysis of the other squares that use the fact that q is a “Taylor–Wiles” prime. In fact, square (a) already expresses that “a representation of $\pi_1(\mathbb{Z}[\frac{1}{S_q}])$ unramified at q is actually a representation of $\pi_1(\mathbb{Z}[\frac{1}{S}])$.” The reason to go to all the trouble of going through squares (b), (c), is that the functors $\mathcal{F}_{\mathbb{Z}_q}$ and $\mathcal{F}_{\mathbb{Q}_q}$ are not representable – they have too many automorphisms. But the framed T -functors,, and of course the trivial functor $*$ and the functor $\mathrm{Hom}_{\mathrm{Art}_k}((S_q^\circ)^c, -)$, are pro-representable.

By concatenating all the squares, we arrive at a homotopy pullback square

$$(8.3) \quad \begin{array}{ccc} \mathcal{F}'_{\mathbb{Z}[\frac{1}{S}]} & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ \mathcal{F}'_{\mathbb{Z}[\frac{1}{S_q}]} & \longrightarrow & \mathrm{Hom}_{\mathrm{Art}_k}((S_q^\circ)^c, -) \end{array}$$

where the functor $\mathcal{F}'_{\mathbb{Z}[\frac{1}{S_q}]}$ comes with a natural weak equivalent to $\mathcal{F}_{\mathbb{Z}[\frac{1}{S_q}]}$, etc.; we have to make this naturally equivalent replacements to “invert” the horizontal weak equivalences of (8.2) (cf. discussion after (7.2)).

If we were dealing with the usual deformation rings, the corresponding statement is (cf. (1.9))

$$(8.4) \quad (\text{deformation ring at level } S) = (\text{deformation ring at level } Sq) \otimes_{S_q^\circ} W(k).$$

The diagram (8.3) will imply a similar result for the derived deformation rings, with tensor products now taken in the derived sense.

Now let us introduce notation for representing rings:

- We denote representing rings for $\mathcal{F}_{\mathbb{Q}_q}^{T, \square}$ and $\mathcal{F}_{\mathbb{Z}_q}^{T, \square}$ by \mathcal{S}_q and $\mathcal{S}_q^{\mathrm{ur}}$ respectively (the superscript “ur” is for “unramified,” and we omit any square \square on the representing rings for typographical simplicity – \mathcal{S} -rings will always denote framed deformations into T .) Thus these are representing rings for the framed deformation functors of $\rho_{\mathbb{Q}_q}^T$ and $\rho_{\mathbb{Z}_q}^T$ with targets in T .

- We set also $S_q = \pi_0 \mathcal{S}_q$ and $S_q^{\text{ur}} = \pi_0 \mathcal{S}_q^{\text{ur}}$. These are, therefore, the (un-derived) framed deformation rings for $\rho_{\mathbb{Q}_q}^T$ and $\rho_{\mathbb{Z}_q}^T$, again with targets in T .
- Recall the definition of $I_q \cong (\mathbb{Z}/q)^*$ from (6.2). The usual (underived) framed deformation ring of the trivial representation: $I_q \longrightarrow T(k)$ will be denoted by S_q° .

The remainder of the section will prove that each square (a), (b), (c), (d) in (8.2) is homotopy pullback, construct the sections s and the map j . We will analyze the squares (a), (b), (c), (d) in order.

8.2. The square (a) from (8.2) is a homotopy pullback square. Examine just square (a):

$$(8.5) \quad \begin{array}{ccc} \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & \mathcal{F}_{\mathbb{Z}_q} \\ \downarrow & (a) & \downarrow \\ \mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]} & \longrightarrow & \mathcal{F}_{\mathbb{Q}_q} \end{array}$$

This arises from a corresponding diagram of pro-simplicial sets, the étale homotopy types of the following diagram

$$(8.6) \quad \begin{array}{ccc} \text{Spec } \mathbb{Q}_q & \longrightarrow & \text{Spec } \mathbb{Z}[\frac{1}{Sq}] \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{Z}_q & \longrightarrow & \text{Spec } \mathbb{Z}[\frac{1}{S}] \end{array}$$

To verify that square (a) is homotopy pullback, it is enough to check that the corresponding map on tangent complexes is also a pullback square (see discussion after Lemma 4.30) that is to say, that the induced map

$$(8.7) \quad \mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \rightarrow \text{hofib}(\mathfrak{t}\mathcal{F}_{\mathbb{Z}_q} \oplus \mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]} \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_q})$$

is an isomorphism. Here the “homotopy fiber” $\text{hofib}(A_{\bullet} \rightarrow B_{\bullet})$ of a map of chain complexes is the chain complex $C_{\bullet} := A_{\bullet} \oplus B_{\bullet}[-1]$, with the usual mapping cone differential; it fits into a triangle $C_{\bullet} \rightarrow A_{\bullet} \rightarrow B_{\bullet} \xrightarrow{[1]}$. In other words (Lemma 5.10) we should get a homotopy pullback square when we apply étale cochains, valued in $\text{Ad}\rho$, to the étale homotopy type of the diagram (8.6).

To verify that, we can replace $\text{Spec } \mathbb{Z}_q$ and $\text{Spec } \mathbb{Q}_q$ by $\text{Spec } \mathbb{Z}_q^{hs}$ and $\text{Spec } \mathbb{Q}_q^{hs}$, where \mathbb{Z}_q^{hs} is the henselization of \mathbb{Z} at the closed point corresponding to q (one can present \mathbb{Z}_q^{hs} as the ring of integers inside the algebraic closure of \mathbb{Q} within \mathbb{Q}_q) and \mathbb{Q}_q^{hs} is its quotient field, obtained by inverting q . In fact the maps $\mathbb{Z}_q^{hs} \rightarrow \mathbb{Z}_q$ and $\mathbb{Q}_q^{hs} \rightarrow \mathbb{Q}_q$ induce isomorphisms in étale cohomology. (Actually this step is wholly cosmetic; we could replace everywhere \mathbb{Z}_q by \mathbb{Z}_q^{hs} .)

In turn the henselization can be presented as a direct limit of the ring of regular functions on étale coverings $V \rightarrow \operatorname{Spec} \mathbb{Z}[\frac{1}{S}]$ equipped with a lift of $\operatorname{Spec} \mathbb{Z}/q \rightarrow \operatorname{Spec} \mathbb{Z}[\frac{1}{S}]$; and the induced map

$$\varinjlim C_{et}^*(V; \operatorname{Ad} \rho) \rightarrow C_{et}^*(\mathbb{Z}_q^{\text{hs}}; \operatorname{Ad} \rho)$$

is a quasi-isomorphism; there is a similar assertion for \mathbb{Q}_q^{hs} replacing V by $V \times_{\mathbb{Z}[\frac{1}{S}]} \mathbb{Z}[\frac{1}{Sq}]$ (this can be seen directly or see [1, VIII, Corollaire 5.8]).

We are reduced in this way to the “Mayer–Vietoris” sequence: if $V \rightarrow X$ is an étale map, and $U \subset X$ is Zariski-open, the corresponding sequence of étale cochains

$$C^*(X) \rightarrow C^*(U) \oplus C^*(V) \rightarrow C^*(U \times_X V)$$

induces an isomorphism from $C^*(X)$ to the homotopy fiber of the latter morphism. After checking the compatibility of connecting homomorphisms, this follows from the cohomological version proved in [32, Tag 0A50]; there both U, V are Zariski-open and $U \times_X V = U \cap V$, but the proof only requires *one* of them to be Zariski-open. We apply this with $X = \operatorname{Spec} \mathbb{Z}[\frac{1}{S}]$, $U = \operatorname{Spec} \mathbb{Z}[\frac{1}{Sq}]$, and V as above. \square

8.3. Analysis of square (b) from diagram (8.2). The following Lemma shows that the horizontal maps of square (b) are weak equivalences:

Lemma 8.3. *Let q be a Taylor–Wiles prime for ρ , as described above. Assume by conjugating ρ that $\rho(\operatorname{Frob}_q)$ lies in $T(k)$. Let $\pi_1 \mathbb{Z}_q$ and $\pi_1 \mathbb{Q}_q$ act on $\operatorname{Lie}(T)_k$ and $\operatorname{Lie}(G)_k$ by means of the adjoint action, composed with ρ . Then all horizontal maps in the following diagram are isomorphisms:*

$$(8.8) \quad \begin{array}{ccc} H^*(\mathbb{Z}_q, \operatorname{Lie}(T)_k) & \longrightarrow & H^*(\mathbb{Z}_q, \operatorname{Lie}(G)_k) \\ \downarrow & & \downarrow \\ H^*(\mathbb{Q}_q, \operatorname{Lie}(T)_k) & \longrightarrow & H^*(\mathbb{Q}_q, \operatorname{Lie}(G)_k) \end{array}$$

For the notation, see the last paragraph of §6.2: in particular $H^*(\mathbb{Z}_q)$ is the étale cohomology of $\operatorname{Spec} \mathbb{Z}_q$, etc.

Proof. Indeed, for H^0 on either the top row or the bottom row, this is just the assertion that the fixed space of $\rho(\operatorname{Frob}_q)$ on $\operatorname{Lie}(G)_k$ is exactly $\operatorname{Lie}(T)_k \subset \operatorname{Lie}(G)_k$, which follows from the assumed (§6.7) strong regularity of $\rho(\operatorname{Frob}_q)$.

Note in what follows that the inclusion $\operatorname{Lie}(T)_k \subset \operatorname{Lie}(G)_k$ is $T(k)$ -equivariantly split (by means of root spaces).

For H^2 , the top row is identically zero: the étale cohomology of \mathbb{Z}_q is in dimensions 0, 1 only. Bijectivity of the bottom map for H^2 amounts (under Poitou–Tate duality) to the fact that Frob_q fixes only $\operatorname{Hom}(\operatorname{Lie}(T)_k, \mu_{p^\infty})$ inside $\operatorname{Hom}(\operatorname{Lie}(G)_k, \mu_{p^\infty})$: this follows from the assumption that $q = 1$ in k , so that the cyclotomic character is trivial, and the assumption of regularity. Note in particular that

$$(8.9) \quad \dim H^2(\mathbb{Q}_q, \operatorname{Lie}(G)_k) = \operatorname{rank} G.$$

It remains to check for H^1 .

On the bottom row: By the Euler characteristic formula the map goes between groups of the same sizes; it is injective because the inclusion of $\pi_1 \mathbb{Q}_q$ -modules $\mathrm{Lie}(T)_k \rightarrow \mathrm{Lie}(G)_k$ is split, so it is an isomorphism.

On the top row: we check root space by root space: our assertion comes down to the triviality of the first group cohomology H^1 , for the profinite group $\hat{\mathbb{Z}}$ acting nontrivially on k . \square

Remark 8.4. *The Lemma implies that any lift of $\rho_{\mathbb{Q}_q} : \pi_1 \mathbb{Q}_q \rightarrow G(k)$ actually factors through $\pi_1 \mathbb{Q}_q^{\mathrm{tame}, \mathrm{ab}}$. Indeed, the Lemma means that the map from the T -valued deformation functor of $\rho_{\mathbb{Q}_q}^T$ to the G -valued deformation functor of $\rho_{\mathbb{Q}_q}^T$ is an equivalence. This implies that any lift of $\rho_{\mathbb{Q}_q}^T$ into G admits a unique T -valued conjugate; in particular, such a lift factors through $\pi_1 \mathbb{Q}_q^{\mathrm{tame}, \mathrm{ab}}$, as claimed.*

Later on we will use the following terminology:

Definition 8.5. *Let q be a Taylor–Wiles prime and A a (usual) Artin local ring over k . A (usual, underived) lift $\tilde{\rho} : \pi_1 \mathbb{Q}_q \rightarrow G(A)$ of $\rho_{\mathbb{Q}_q}$ is of inertial level $\leq r$ if the restriction of $\tilde{\rho}$ to I_q :*

$$(8.10) \quad I_q \rightarrow G(A)$$

factors through I_q/p^r . Here $I_q \leq \pi_1 \mathbb{Q}_q^{\mathrm{tame}, \mathrm{ab}}$ is as in (6.2).

Thus, for example, if the p -valuation of $q - 1$ is exactly equal to n , then every deformation of $\rho_{\mathbb{Q}_q}$ is certainly of inertial level $\leq n$, because in that case the p -part of I_q has size precisely p^n .

8.4. Analysis of square (c) from diagram (8.2). Construction of the sections $s_{\mathbb{Z}_q}, s_{\mathbb{Q}_q}$ in the diagram (8.2). We now examine the square (c), which we draw at first without the splittings:

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{Z}_q}^T & \longleftarrow & \mathcal{F}_{\mathbb{Z}_q}^{T, \square} \\ \downarrow & (c) & \downarrow \\ \mathcal{F}_{\mathbb{Q}_q}^T & \longleftarrow & \mathcal{F}_{\mathbb{Q}_q}^{T, \square} \end{array}$$

This is a homotopy pullback square: both above and below, one can obtain the framed version from the unframed version by taking a homotopy fiber of a map to $BT(A)$.

We will construct splittings for

$$\mathcal{F}_{\mathbb{Q}_q}^T \leftarrow \mathcal{F}_{\mathbb{Q}_q}^{T, \square}$$

and the \mathbb{Z}_q analog, compatibly. Actually, strictly speaking, the splitting we construct is in the sense of zig-zags; explicitly we construct functors $(\dots)^*$ naturally

weakly equivalent to the corresponding unstarred functors (\dots) and a diagram

$$(8.11) \quad \begin{array}{ccccc} \mathcal{F}_{\mathbb{Z}_q}^{T,*} & \longrightarrow & \mathcal{F}_{\mathbb{Z}_q}^{T,\square,*} & \longrightarrow & \mathcal{F}_{\mathbb{Z}_q}^{T,*} \\ \downarrow & & \downarrow & (c)^* & \downarrow \\ \mathcal{F}_{\mathbb{Q}_q}^{T,*} & \longrightarrow & \mathcal{F}_{\mathbb{Q}_q}^{T,\square,*} & \longrightarrow & \mathcal{F}_{\mathbb{Q}_q}^{T,*} \end{array}$$

where the horizontal compositions are the identity, and the square $(c)^*$ is connected by natural weak equivalences of squares, to square (c). It then follows that the left-hand square is also a homotopy pullback square.

To make this splitting, we note that the A -points of framed and unframed representation functors into T correspond to based and unbased maps of (the étale homotopy type of) $\mathrm{Spec} \mathbb{Q}_q$ into $BT(A)$. Since T is commutative, it is possible to make a weakly equivalent model $BT(A)^*$ of $BT(A)$ which is a simplicial group (for example, take the simplicial loop group of $B^2T(A)$; the construction of $B^2T(A)$ is discussed after Definition 5.1).

Now, for \mathcal{E} a simplicial group and X an arbitrary based simplicial set the natural map

$$\mathrm{based} \mathrm{Hom}_{s\mathrm{Sets}}(X, \mathcal{E}) \rightarrow \mathrm{Hom}_{s\mathrm{Sets}}(X, \mathcal{E})$$

is naturally split, because one can use the group structure to take the image of the basepoint in X to the identity. This discussion gives rise to a splitting of the map $(\mathcal{F}_{X,T}^{\square})^* \rightarrow (\mathcal{F}_{X,T})^*$ (notation as in Definition 5.4) whenever G is commutative; we have superscripted with $*$ because we replaced BT by $(BT)^*$ in the definitions.

Finally, the actual deformation functors are obtained from these (up to replacement by a naturally weakly equivalent functor) by taking a homotopy fiber over a certain vertex $v_\rho \in (\mathcal{F}_{X,T}^{\square})^*(k)$ and its image $\bar{v}_\rho \in \mathcal{F}_{X,T}^*(k)$; we may choose any vertex v_ρ whose class in $\pi_0((\mathcal{F}_{X,T}^{\square})^*(k)) \simeq \pi_0(\mathcal{F}_{X,T}(k))$ is the class corresponding to $\pi_1(X, x_0) \rightarrow T(k)$.

8.5. Analysis of the square (d); Construction of the map j . A very useful fact is that the pro-rings S_q^{ur} and S_q representing the functors $\mathcal{F}_{\mathbb{Z}_q}^{T,\square}$ and $\mathcal{F}_{\mathbb{Q}_q}^{T,\square}$ are already homotopy discrete:

Lemma 8.6. *Let q be a Taylor–Wiles prime (§6.7). The pro-rings S_q^{ur} and S_q are homotopy discrete in the sense of Definition 7.4, i.e. the maps $S_q^{\mathrm{ur}} \rightarrow S_q^{\mathrm{ur}}$ and $S_q \rightarrow S_q$ induce a weak equivalence of represented functors (as usual, after applying level-wise cofibrant replacement).*

A word of preparation before the proof. Any (usual, underived) T -valued deformation of $\rho_{\mathbb{Q}_q}^T$ factors through the tame abelian quotient of $\pi_1 \mathbb{Q}_q$, which fits into the exact sequence (6.2); we have, non-canonically,

$$(8.12) \quad \pi_1 \mathbb{Q}_q^{\mathrm{tame,ab}} \cong \langle \mathrm{Frob}_q \rangle \times I_q$$

where I_q is as in (6.2), a cyclic group of order $q - 1$.

Proof (of the Lemma). We apply Lemma 7.5. It is enough to check that the complete local rings associated to S_q^{ur} and S_q (as in Lemma 7.5, the inverse limit of the corresponding projective system) are complete intersection “of the expected size,” i.e. their tangent spaces and dimensions are as one would predict from the tangent complex.

By a mild abuse of terminology, in the remainder of this proof, we will use S_q^{ur} and S_q to actually denote these complete local rings. We emphasize that in the rest of the proof we are working with underived deformation rings.

It will be convenient to recall the following simple fact: For a finitely generated discrete group Γ , with profinite completion $\hat{\Gamma}$, the usual (underived) framed deformation ring of a representation $\rho_0 : \hat{\Gamma} \rightarrow G(k)$ is (after passing to the associated complete ring) the completed local ring $\hat{\mathcal{O}}_{X, \rho_0}$ of the $W(k)$ -scheme X parameterizing maps $\Gamma \rightarrow G$, at the point $\rho_0 \in X(k)$.

First of all, S_q^{ur} is a power series ring over $W(k)$ because it arises from the deformation ring of the pro-cyclic group $\hat{\mathbb{Z}}$. It obviously has the “expected size.”

Now for S_q . Let p^N be the highest power of p dividing $q - 1$. It follows from (8.12) that S_q is the completed local ring of functions on

$$\left((F, t_2) \in T \times T : t_2^{p^N} = 1 \right)$$

at the maximal ideal corresponding to $F = \rho(\text{Frob}_q)$ and $t_2 = \text{identity}$. In other words, if A is the completion of ring of functions on $(t_1, t_2) \in T \times T$ at the identity point of $T(k) \times T(k)$, and J the ideal defined by $t_2^{p^N} = e$, we have

$$S_q \cong A/J.$$

In suitable coordinates, with $r = \dim(T)$, we have

$$(8.13) \quad A \cong W[[X_1, \dots, X_r, Y_1, \dots, Y_r]], \quad J = \langle (1 + Y_i)^{p^N} - 1 \rangle$$

This is visibly a complete intersection, with r relations; and the dimension of $\mathfrak{t}^1 \mathcal{F}_{S_q}$ equals $\dim H^2(\mathbb{Q}_q, \text{Ad} \rho) = r$ by (8.9). \square

We are now ready to construct the final square (d):

Clearly, any T -valued deformation $\tilde{\rho}$ of $\rho|_{\mathbb{Q}_q^T}$ actually factors through $\pi_1(\mathbb{Q}_q)^{\text{tame, ab}}$.

Thus, from the sequence (6.2) $I_q \hookrightarrow \pi_1 \mathbb{Q}_q^{\text{tame, ab}} \rightarrow \pi_1 \mathbb{F}_q$ we get a sequence of maps of the associated T -valued framed usual deformation rings

$$(8.14) \quad \underbrace{S_q^\circ}_{W[[Y_1, \dots, Y_r]]/J} \rightarrow \underbrace{S_q}_{W[[X_1, \dots, X_r, Y_1, \dots, Y_r]]/J} \rightarrow \underbrace{S_q^{\text{ur}}}_{W[[X_1, \dots, X_r]]}$$

Here, recall that S_q° was the framed deformation ring for the trivial representation of I_q with targets in T ; intrinsically, we can identify S_q° with the group algebra of $\mathbf{T}(\mathbb{F}_q)_p$. The maps of (8.14) are really maps of pro-finite rings, but we wrote below how the sequence of associated complete local rings looks in the coordinates of (8.13); the second map sends Y_i to 0.

Remark 8.7. *It is convenient to identify these rings more canonically. For any finitely generated abelian group Γ , the usual, un-derived framed deformation ring of a representation $\sigma : \Gamma \rightarrow T(k)$ is identified with the completed group algebra*

$$(8.15) \quad W(k)[[X_*(\mathbf{T}) \otimes \Gamma_{(p)}]]$$

where $\Gamma_{(p)}$ is the quotient of Γ by prime-to- p torsion.

In fact, the unique splitting of $T(W(k)) \rightarrow T(k)$ gives us a lift $\tilde{\sigma} : \Gamma \rightarrow T(W)$, which permits us to reduce to the case (by twisting) when σ is trivial. Then, for any Artin ring $A \twoheadrightarrow k$, the homomorphisms from Γ to $T(A)$, reducing to the trivial homomorphism, are identified with group homomorphisms

$$\mathrm{Hom}(\Gamma_{(p)}, X_*(T) \otimes (1 + \mathfrak{m}_A)^\times) = \mathrm{Hom}(X^*(T) \otimes \Gamma_{(p)}, 1 + \mathfrak{m}_A),$$

i.e. homomorphisms from (8.15) to A preserving augmentations to k .

In this way, writing \mathbf{T} for the dual torus to T ,

$$S_q^\circ = \text{completed group algebra of } \mathbf{T}(\mathbb{F}_q)$$

$$S_q = \text{completed group algebra of } \mathbf{T}(\mathbb{Q}_q)^{\mathrm{tame}},$$

$$S_q^{\mathrm{ur}} = \text{completed group algebra of } \mathbf{T}(\mathbb{Q}_q)^{\mathrm{ur}}$$

Here the “tame” quotient of, e.g. $\mathbb{G}_m(\mathbb{Q}_q) = \mathbb{Q}_q^$ is the profinite completion of the quotient by $1 + q\mathbb{Z}_q$, and the “unramified” quotient of the same group is the profinite completion of the quotient by \mathbb{Z}_q^* .*

In fact the following diagram

$$(8.16) \quad \begin{array}{ccc} S_q^\circ & \longrightarrow & S_q \\ \downarrow & & \downarrow \\ W(k) & \longrightarrow & S_q^{\mathrm{ur}} \end{array}$$

induces a homotopy pullback square of represented functors (equivalently the map $S_q \underline{\otimes}_{S_q^\circ} W(k) \rightarrow S_q^{\mathrm{ur}}$, where $\underline{\otimes}$ is the derived tensor product, is an equivalence, in the sense of inducing a weak equivalence on represented functors). This can be checked easily from the expressions above: S_q^{ur} is obtained from $W(k)$ by freely adding the generators X_i , and S_q is obtained from S_q° by the same procedure.

Passing from (8.16) to the corresponding square of represented functors, and using Lemma 8.6, we get square (d) from (8.2). As we have mentioned in Remark 8.1, this can strictly speaking only be done after replacing the functors with weakly equivalent ones, for the reasons already mentioned there.

Remark: In our later presentation of the Taylor–Wiles method, it will be useful to observe that exactly the same holds true if we quotient all the rings by p^n , that

is to say, writing $\overline{S}_q^\circ = S_q^\circ/p^n$.

$$(8.17) \quad \begin{array}{ccc} \overline{S}_q^\circ & \longrightarrow & S_q/p^n \\ \downarrow & & \downarrow \\ W_n & \longrightarrow & S_q^{\text{ur}}/p^n \end{array}$$

again induces a homotopy pullback square of represented functors.

9. THE DEFORMATION RING WITH LOCAL CONDITIONS IMPOSED

In the theory of Galois deformations much of the subtlety comes from the local theory at p . Thus we discuss how to impose local conditions on a derived deformation ring. After an abstract discussion in §9.1, we specify which conditions we actually use in §10.

9.1. Imposing local conditions. Let $\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$. As discussed, for v a finite place⁶ in S we let $\mathcal{F}_{\mathbb{Q}_v}$ be the deformation functor for ρ pulled back to \mathbb{Q}_v ; thus, there is a natural transformation

$$\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \rightarrow \mathcal{F}_{\mathbb{Q}_v}$$

of functors from Art_k to $s\text{Sets}$. Note that $\mathcal{F}_{\mathbb{Q}_v}$ is often not representable because of infinitesimal automorphisms, even when $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}$ is, but it doesn't matter for our discussion.

Definition 9.1. A local deformation condition will be, by definition, a simplicially enriched functor \mathcal{D}_v from Art_k to $s\text{Sets}$, equipped with a map

$$\mathcal{D}_v \longrightarrow \mathcal{F}_{\mathbb{Q}_v}$$

We define the global deformation functor with local conditions as the homotopy fiber product (see §A.3)

$$(9.1) \quad \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^{\mathcal{D}} = \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \times_{\mathcal{F}_{\mathbb{Q}_v}}^h \mathcal{D}_v.$$

This comes with natural maps to \mathcal{D}_v and to $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}$.

Remark. It will happen (indeed, in the main case we use it) that a deformation condition is not presented as a map $\mathcal{D}_v \longrightarrow \mathcal{F}_{\mathbb{Q}_v}$, but rather as a zig-zag:

$$(9.2) \quad \mathcal{D}_v \xleftarrow{\sim} \mathcal{D}_v^* \rightarrow \mathcal{F}_{\mathbb{Q}_v}^* \xleftarrow{\sim} \mathcal{F}_{\mathbb{Q}_v},$$

where \sim denotes an object-wise weak equivalence. In this case, we convert this to a local deformation condition in the sense above by setting $\mathcal{D}'_v = \mathcal{D}_v^* \times_{\mathcal{F}_{\mathbb{Q}_v}^*}^h \mathcal{F}_{\mathbb{Q}_v}$; then \mathcal{D}'_v is naturally weakly equivalent to \mathcal{D}_v , and is equipped with a map $\mathcal{D}'_v \rightarrow \mathcal{F}_{\mathbb{Q}_v}$. One can proceed in a similar way with longer zig-zags; in all cases one

⁶When we will apply this discussion later on, we will first have enlarged S to a larger set $S' = S \amalg Q$, and we will actually take $v \in Q$.

obtains a functor naturally weakly equivalent to \mathcal{D}_v with an actual map to $\mathcal{F}_{\mathbb{Q}_v}$. Cf. discussion around (7.2).

Example (unramified local condition): Suppose that ρ is actually unramified at v , i.e., it extends to a representation of $\mathbb{Z}[\frac{1}{S'}]$, where $S' = S - \{v\}$. Thus it can be pulled back to $\text{Spec } \mathbb{Z}_v$. The natural maps $\text{Spec } \mathbb{Q}_v \rightarrow \text{Spec } \mathbb{Z}_v$ induces a map of deformation functors $\mathcal{F}_{\mathbb{Z}_v} \rightarrow \mathcal{F}_{\mathbb{Q}_v}$ (as always, we are deforming ρ , but we omit this from our notation). If we take

$$\mathcal{D}_v = \mathcal{F}_{\mathbb{Z}_v}, \text{ with its natural map to } \mathcal{F}_{\mathbb{Q}_v},$$

the corresponding deformation functor with local conditions $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^{\mathcal{D}}$ is then naturally weakly equivalent to $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}$, i.e. the deformations of ρ but considered now as a representation of $\pi_1 \mathbb{Z}[\frac{1}{S'}]$. This statement is precisely what we proved in the last section: it follows from the homotopy pullback square constructed in §8.2 (the assumption of q being a Taylor-Wiles prime was not used for this part).

Return now to the general case. We get from Lemma 4.30

$$(9.3) \quad \mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D = \text{hofib}(\mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \oplus \mathfrak{t}\mathcal{D}_v \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_v})$$

(for hofib see (8.7)) with associated exact sequence

$$(9.4) \quad \mathfrak{t}^i \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D \rightarrow \mathfrak{t}^i \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \oplus \mathfrak{t}^i \mathcal{D}_v \rightarrow \mathfrak{t}^i \mathcal{F}_{\mathbb{Q}_v} \xrightarrow{[1]}$$

One can generalize this definition and discussion, replacing the role of v by a finite set of places.

9.2. Lifting an underived local condition to a derived local condition. In [7, §2] there is an extensive study of various types of (underived) local conditions D . Each such gives rise to a quotient of the usual (underived) framed deformation ring $R_v^{\square} \twoheadrightarrow R_v^{D, \square}$; if the representation functor is representable, we get in fact a quotient of the underived deformation ring:

$$R_v \twoheadrightarrow R_v^D.$$

(These objects are actually pro-rings; we use the notation \twoheadrightarrow to denote that the induced map of functors is injective, or equivalently that the map on associated complete local rings is surjective.)

Assume, for simplicity, that $\mathcal{F}_{\mathbb{Q}_v}$ is representable, with representing ring \mathcal{R}_v ; then $\pi_0 \mathcal{R}_v \simeq R_v$. We can lift an underived local deformation condition to a derived deformation condition, in the sense of §9.1, by using the sequence (for the framed case, see below):

$$\mathcal{R}_v \rightarrow \pi_0 \mathcal{R}_v = R_v \twoheadrightarrow R_v^D.$$

More precisely, considering now R_v^D as a pro-simplicial ring, we may form $\mathcal{D}_v = \text{Hom}((R_v^D)^c, -)$, as a $s\text{Sets}$ -valued functor on Art_k (the superscript c is

for level-wise cofibrant replacement, as discussed in the final paragraph of §7.1). Then we get an natural zig-zag:⁷

$$(9.5) \quad \mathcal{D}_v \dashrightarrow \mathrm{Hom}(\mathcal{R}_v, -)$$

As in the Remark of §9.1 we obtain from (9.5) a map

$$(9.6) \quad \mathcal{D}_v^* \longrightarrow \mathcal{F}_{\mathbb{Q}_v},$$

where \mathcal{D}_v^* is naturally weakly equivalent to $\mathrm{Hom}((R_v^D)^c, -)$. We call this the *derived deformation condition associated with the (usual) deformation condition D*.

Before we formulate our main theorem, note that $R_v \twoheadrightarrow R_v^D$ defines a subspace $H_D^1 \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\rho)$, where H_D^1 is the (usual) tangent space to the (usual) functor represented by R_v^D .

Theorem 9.2. *Suppose that R_v^D is actually formally smooth (i.e. its tangent complex is nonvanishing only in degree 0, or equivalently the complete local ring associated to R_v^D is isomorphic to $W(k)[[Y_1, \dots, Y_m]])$. Form from R_v^D a local deformation condition, as described above, and then a global deformation functor $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D$ with this local condition imposed, as in (9.1).*

The cohomology of the tangent complex of $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D$ is naturally identified with the cohomology with local conditions (§6.3):

$$(9.7) \quad \mathfrak{t}^i \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D \cong H_D^{i+1}(\mathbb{Z}[1/S], \mathrm{Ad}\rho).$$

where the local conditions at v are prescribed by the subspace $H_D^1 \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\rho)$.

Proof. Consider the map of tangent complexes $\mathfrak{t}\mathcal{D}_v \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}$. The canonical map $\tau_{\geq 0}(\mathfrak{t}\mathcal{D}_v) \rightarrow \mathfrak{t}\mathcal{D}_v$ is a quasi-isomorphism; and so the tangent complex of $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D$ is naturally equivalent to the homotopy fiber of

$$\mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D \oplus \tau_{\geq 0}\mathfrak{t}\mathcal{D}_v \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}.$$

However, the map $\tau_{\geq 0}(\mathfrak{t}\mathcal{D}_v) \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}$ can be factored

$$\tau_{\geq 0}(\mathfrak{t}\mathcal{D}_v) \xrightarrow{\varpi} \tau_{\geq 0}(\mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}) \rightarrow \mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}$$

The source and target of the first map ϖ both have homotopy only in degree 0; thus, ϖ induces a quasi-isomorphism onto the subcomplex of $\tau_{\geq 0}\mathfrak{t}\mathcal{F}_{\mathbb{Q}_v}$ corresponding to the subgroup $H_D^1 \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\rho) \cong \pi_0(\tau_{\geq 0}\mathfrak{t}\mathcal{F}_{\mathbb{Q}_v})$.

This discussion gives a zig-zag of weak equivalences (i.e. quasi-isomorphisms) between the tangent complex of $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^D$, and the chain complex (see (B.1)) that computes H_D^i . \square

Remark. When we will use this construction, $\mathcal{R}_v \simeq \pi_0 \mathcal{R}_v$ will be homotopy discrete, and so both \mathcal{R}_v and R_v^D will actually be formally smooth. The definition

⁷Explication: Apply the cofibrant replacement to the series of rings to get

$$\mathcal{R}_v \xleftarrow{\sim} \mathcal{R}_v^c \rightarrow R_v^c \rightarrow R_v^{D,c}$$

where the left \sim means level-wise weak equivalence. Now apply Hom .

makes sense without the assumption that \mathcal{R}_v is homotopy discrete but it seems unlikely to be of much use if this fails.

Remark. If the local representation functors are not representable, we can proceed in a similar fashion using the framed functors instead. Namely, the usual framed deformation ring R_v^\square is equipped with a G -action, i.e.

$$R_v^\square \rightarrow R_v^\square \hat{\otimes} \mathcal{O}_G,$$

where the $\hat{\otimes}$ denotes completed tensor product, i.e. the level-wise tensor product for pro-objects; also, given a quotient $R_v^\square \twoheadrightarrow R_v^{\square, D}$ together with a compatible G -action, we may then form a derived deformation condition by using – just as in §5.1 – a bar construction to “quotient by G .”

10. RESTRICTIONS ON ρ

Recall that we have been discussing the deformation theory of a Galois representation $\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$. Thus far, ρ has been quite general. Now, to simplify our life as far as possible, we will impose the following conditions: (Recall the shorthand that $T = S - \{p\}$, where p is the characteristic of k .)

Assumption 1. *Assumptions on ρ :*

- (a) $H^0(\mathbb{Q}_p, \text{Ad}\rho_{\mathbb{Q}_p}) = H^2(\mathbb{Q}_p, \text{Ad}\rho_{\mathbb{Q}_p}) = 0$.
- (b) For $v \in T$, the local cohomology $H^j(\mathbb{Q}_v, \text{Ad}\rho_{\mathbb{Q}_v}) = 0$ for $j = 0, 1, 2$.
- (c) ρ has big image: the image of ρ restricted to $\mathbb{Q}(\zeta_{p^\infty})$ contains the image of $G^{\text{sc}}(k)$ in $G(k)$ (here G^{sc} is the simply connected cover).
Moreover, we suppose that $G^{\text{sc}}(k)$ has no invariants in the adjoint action.⁸
- (d) ρ_p (the restriction of ρ to $G_{\mathbb{Q}_p}$) satisfies (c) and (e) from Conjecture 6.1 – in particular, it is equipped with a sub-functor $\text{Def}_{\rho_p}^{\text{crys}}$ of the usual deformation functor, the “crystalline deformations,” whose tangent space is the local f -cohomology.

In particular, (a) means that the local deformation ring at p is formally smooth, isomorphic to $W(k)[[Y_1, \dots, Y_s]]$ for $s = \dim H^1(\mathbb{Q}_p, \text{Ad}\rho_{\mathbb{Q}_p})$; and the local deformation ring at v , for $v \in T$, is even isomorphic to $W(k)$. We will sometimes refer to these conditions as “minimal level”, although they are a bit stronger than the usual usage of that term.

Now, in this context, we may use $\text{Def}_{\rho_p}^{\text{crys}}$ and the process of §9.2 to *define* a derived deformation condition at p , and then we get as in Definition 9.1 a global deformation functor $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^{\text{crys}}$ with crystalline conditions imposed. So by (9.7) and the definition (6.5) of the global f -cohomology we obtain

$$(10.1) \quad \mathfrak{t}^i \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]}^{\text{crys}} \cong H_f^{i+1}(\mathbb{Z}[\frac{1}{S}], \text{Ad}\rho).$$

⁸This is automatic, by results of Steinberg [33], if p is a very good prime for \mathbf{G} ; in particular if $p > 5$ and does not divide $r + 1$ for any \mathfrak{sl}_r factor of the Lie algebra, cf. [19, 6.4(b)].

The functor and its representing ring will be our main object of study in the rest of this paper:

In the rest of the paper, whenever we refer to a representation $\rho : \Gamma_S \rightarrow G(k)$, we assume that it satisfies conditions (a)–(d) above; whenever we refer to a deformation ring or deformation functor for ρ , considered as a global Galois representation, we always understand the “crystalline deformation ring” or “crystalline deformation functor” – that is to say, we have imposed crystalline local conditions at p , in the sense just described. We henceforth drop the subscript “crys” from the notation.

Remark. Why should the “lifting” process of §9.2 give a reasonable definition of the derived version of the crystalline deformation functor? In the Fontaine–Laffaille range, one can reasonably guess that the tangent space of a putative derived version of the local crystalline functor should be given by the f -cohomology (§6.4) and this is enough to force this functor to be formally smooth, thus with representing ring that is homotopy discrete. Therefore, it must arise by means of the construction of §9.2.

11. DESCENDING FROM TAYLOR–WILES LEVEL

We continue with a representation $\rho : \Gamma_S \rightarrow G(k)$ satisfying the assumptions fixed in §10. It is ramified at a set $S = T \cup \{p\}$. We denote by \mathcal{R}_S the crystalline deformation ring of ρ ; see §7.4 for a summary of our various notations concerning deformation functors and rings.

In the Taylor–Wiles method, one studies \mathcal{R}_S by relating it to a larger deformation ring, wherein one allows extra ramification. Namely, we consider the deformation ring which also allows ramification at an auxiliary set Q_n of primes, which is allowable in the sense of Definition 6.2 – i.e., Q_n satisfies some carefully chosen cohomological criteria. Although one eventually uses a sequence of such sets of auxiliary primes (thus the notation Q_n), *the set Q_n of auxiliary primes can be regarded as fixed in this section*. It is only in later sections that we will study the situation as one varies the set of auxiliary primes.

We will study the deformation rings further under these cohomological conditions. More precisely, we have seen in §8 – see especially (8.4) – how to recover the derived deformation ring at base level from the derived deformation ring at level SQ_n . Here we will see that the derived deformation ring at level S can actually be “well-approximated,” at least as far as t^0 and t^1 go, just using the *usual* deformation ring at level SQ_n , or even a sufficiently deep Artinian quotient of it; the final result is Theorem 11.1.

11.1. Review of tensor products. In Definition 3.3 we have given a definition of the derived tensor products $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{C}$ of pro-simplicial rings. This definition is somewhat *ad hoc*, and not even entirely functorial; however, it is functorial for diagrams with a common level representation (i.e., where all of the pro-objects are indexed by the same category), which will be enough for our needs; we will often

briefly abusively still say “by functoriality.” The derived tensor product also has the following property (see Definition 3.3 and the discussion following it):

Suppose given functors $\mathcal{F}_{\mathcal{A}} \rightarrow \mathcal{F}_{\mathcal{B}} \leftarrow \mathcal{F}_{\mathcal{C}}$ and representing pro-rings $\mathcal{A}, \mathcal{B}, \mathcal{C}$ which are *nice* in the sense of Definition 2.23. According to Lemma 2.25 we may promote the functor maps to ring maps (i.e., maps in pro-Art_k) $\mathcal{A} \leftarrow \mathcal{B} \rightarrow \mathcal{C}$. In this case

$$(11.1) \quad \mathcal{A} \otimes_{\mathcal{B}} \mathcal{C} \text{ pro-represents the functor } \mathcal{F}_{\mathcal{A}} \times_{\mathcal{F}_{\mathcal{B}}}^h \mathcal{F}_{\mathcal{C}}.$$

Although this looks obvious, this requires a word of explanation to navigate the various homotopies, so let us talk through the technical details:

Denote by (e.g.) $\text{Hom}'(\mathcal{A}, -)$ the mapping space defined as in (7.5) but replacing colimit by homotopy colimit. Lemma 2.25 gives us a diagram

$$(11.2) \quad \begin{array}{ccccc} \mathcal{F}_{\mathcal{A}} & \longrightarrow & \mathcal{F}_{\mathcal{B}} & \longleftarrow & \mathcal{F}_{\mathcal{C}} \\ \uparrow & & \uparrow & & \uparrow \\ \text{Hom}'(\mathcal{A}, -) & \longrightarrow & \text{Hom}'(\mathcal{B}, -) & \longleftarrow & \text{Hom}'(\mathcal{C}, -) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}(\mathcal{A}, -) & \longrightarrow & \text{Hom}(\mathcal{B}, -) & \longleftarrow & \text{Hom}(\mathcal{C}, -) \end{array}$$

The squares commute only up to natural simplicial homotopy, as in Lemma 2.25. By inserting some extra weak equivalences, we may replace this diagram with a strictly commutative diagram: given a “homotopy coherent” collection of maps $X_i \rightarrow Y_i$ indexed by a diagram I (in our case, the arrows along a row) we replace it by the diagram $X_i \xleftarrow{\sim} X'_i \rightarrow Y_i$, where we may take $X'_i = \text{hocolim}_{j \rightarrow i} X_j$ (see e.g. [31, §10]). Thus we have enlarged the above diagram, by inserting various weak equivalences, to obtain a strictly commutative diagram. Take homotopy limits of each row to obtain a zig-zag of weak equivalences

$$\text{Hom}(\mathcal{A}, -) \times_{\text{Hom}(\mathcal{B}, -)}^h \text{Hom}(\mathcal{C}, -) \xrightarrow{\sim} \mathcal{F}_{\mathcal{A}} \times_{\mathcal{F}_{\mathcal{B}}}^h \mathcal{F}_{\mathcal{C}},$$

and this gives a zig-zag of equivalences exhibiting (11.1).

11.2. Sets of Taylor–Wiles primes. Let Q_n be an allowable Taylor–Wiles datum of level n , as in §6.7. We’ll extend the functor notation of §8.1 to many primes thus:

$\mathcal{F}_{S \coprod Q_n}$ or \mathcal{F}_n when clear = crystalline deformation functor at level $S \coprod Q_n$

$\mathcal{R}_{S \coprod Q_n}$ or \mathcal{R}_n = representing ring for $\mathcal{F}_{S \coprod Q_n}$

$$\mathcal{F}_n^{\text{loc}} = \prod_{q \in Q_n} \mathcal{F}_{\mathbb{Q}_q}^{T, \square} = \text{“framed def. functor at primes in } Q \text{ into the torus”}$$

\mathcal{S}_n = representing ring for $\mathcal{F}_n^{\text{loc}}$

$$\mathcal{F}_n^{\text{loc, ur}} = \prod_{q \in Q} \mathcal{F}_{\mathbb{Z}_q}^{T, \square} = \text{“unramified framed def. functor at primes in } Q \text{ into the torus”}$$

$\mathcal{S}_n^{\text{ur}}$ = representing ring for $\mathcal{F}_n^{\text{loc, ur}}$

We apologize for one feature of the notation: the \mathcal{S} -rings and the \mathcal{F}^{loc} -functors are framed, but we have not put explicit \square into the notation, to keep the typography simple.

Using squares (a), (b), (c) of (8.2), or rather an analogous diagram imposing crystalline conditions at each stage and using Q_n instead of just $\{q\}$, gives

$$(11.3) \quad \mathcal{F}'_S \coprod_{Q_n} \times_{\mathcal{F}_n^{\text{loc}'}}^h \mathcal{F}_n^{\text{loc}, \text{ur}'} \xrightarrow{\sim} \mathcal{F}_S$$

where a prime denotes a weakly equivalent functor.

The functors on the left are pro-representable, with representing pro-rings $\mathcal{R}_n, \mathcal{S}_n, \mathcal{S}_n^{\text{ur}}$ (by definition, if a functor is pro-representable, so is a weakly equivalent functor, and they can be taken to be represented by the same ring.) We may suppose that these representing rings are nice, in the sense of Definition 2.23. Just as in the discussion of §11.1, we obtain a corresponding diagram of rings $\mathcal{R}_n \leftarrow \mathcal{S}_n \rightarrow \mathcal{S}_n^{\text{ur}}$.

The maps $\mathcal{F}'_S \coprod_{Q_n} \rightarrow \mathcal{F}_n^{\text{loc}'}$ gives a map $\mathcal{S}_n \rightarrow \mathcal{R}_n$, by Lemma 2.25 and the Remark following it; it is compatible with the original map of functors in the sense specified in that Lemma. The corresponding map $\pi_0 \mathcal{S}_n \rightarrow \pi_0 \mathcal{R}_n$ is then the natural map of usual (underived) deformation rings. Similarly we have $\mathcal{S}_n \rightarrow \mathcal{S}_n^{\text{ur}}$.

The equivalence (11.3) implies that

$$(11.4) \quad \mathcal{R}_S \simeq \mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}}$$

where \simeq here means that the functors that they represent are naturally weakly equivalent. A brief word of warning about this notation is in order. We have not defined “directly” the notion of weak equivalence for pro-simplicial rings, and nor do we need it: we think only in terms of functors represented. However, Lemma 2.25 shows at least that we may find a map of pro-objects $\mathcal{R}_S \rightarrow \mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}}$ which induces an isomorphism on tangent complexes; the assumptions of that Lemma are satisfied because we built “niceness” into the definition of derived tensor product.

11.3. Setup. In what follows we suppose given pro-Artinian quotients

$$(11.5) \quad \pi_0 \mathcal{R}_n \twoheadrightarrow \overline{\mathcal{R}}_n, \pi_0 \mathcal{S}_n \twoheadrightarrow \overline{\mathcal{S}}_n, \pi_0 \mathcal{S}_n^{\text{ur}} \twoheadrightarrow \overline{\mathcal{S}}_n^{\text{ur}}$$

which are compatible, in that there is a diagram $\overline{\mathcal{R}}_n \leftarrow \overline{\mathcal{S}}_n \rightarrow \overline{\mathcal{S}}_n^{\text{ur}}$ which is compatible with the same diagram for the π_0 rings.

We get

$$(11.6) \quad \mathcal{R}_S \xrightarrow{(11.4)} \mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}} \rightarrow \overline{\mathcal{R}}_n \otimes_{\overline{\mathcal{S}}_n} \overline{\mathcal{S}}_n^{\text{ur}}.$$

where \otimes is derived tensor product. Recall that derived tensor product was not genuinely functorial: it is only so when we have fixed a common level representation for all the pro-objects appearing. However, we will allow ourselves to ignore this issue. For one thing, it is easy to fix such a level representation in the case at hand. For another, one could avoid the language of derived tensor product of rings entirely and just work with the functors; we talk about rings just in the hope that they

are easier to relate to. At the level of represented functors, (11.6) corresponds to the diagram (see (7.6) for an explication of this)

$$(11.7) \quad \mathrm{Hom}(\overline{R}_n^c, -) \times_{\mathrm{Hom}(\overline{S}_n^c, -)}^h \mathrm{Hom}((\overline{S}_n^{\mathrm{ur}})^c, -) \dashrightarrow (\mathcal{F}'_S \amalg_{Q_n}) \times_{(\mathcal{F}_n^{\mathrm{loc}})' } (\mathcal{F}_n^{\mathrm{loc}, \mathrm{ur}})' \dashrightarrow \mathcal{F}_S$$

where the second map came from (11.3).

We show that, for allowable Taylor-Wiles data Q_n (cf. §6.7), the map of (11.6) (equivalently the composite of (11.7)) is isomorphic on \mathfrak{t}^0 and surjective on \mathfrak{t}^1 . The point of this result is that the Taylor-Wiles limit process will give very tight control on \overline{R}_n and the other rings appearing on the right-hand side of (11.6).

Theorem 11.1. *Let $Q_n = \{\ell_1, \dots, \ell_q\}$ be an allowable Taylor-Wiles datum of level n , in the sense defined in §6.7. Let other notation be as above. If the maps of (11.5) all induce isomorphisms on \mathfrak{t}^0 , then the map $\mathcal{R}_S \rightarrow \overline{R}_n \otimes_{\overline{S}_n} \overline{S}_n^{\mathrm{ur}}$ (or equivalently the map (11.7) of functors) induces an isomorphism on \mathfrak{t}^0 and a surjection on \mathfrak{t}^1 .*

Proof. Recall our notation for places: p is the characteristic of k , S is the ramification set of ρ , $T = S - \{p\}$, Q is the set of Taylor-Wiles primes. We also put $S' = S \cup Q$. Our “allowability” condition on Taylor-Wiles primes means (see (6.8) and the conditions on ρ from §10) that we get

$$(11.8) \quad \begin{aligned} H^1(\mathbb{Z}[\frac{1}{S'}], \mathrm{Ad}\rho) &\xrightarrow{A} \frac{H^1(\mathbb{Q}_p, \mathrm{Ad}\rho)}{H_f^1(\mathbb{Q}_p, \mathrm{Ad}\rho)}, \\ H^2(\mathbb{Z}[\frac{1}{S'}], \mathrm{Ad}\rho) &\xrightarrow{\sim} \bigoplus_{S'} H^2(\mathbb{Q}_v, \mathrm{Ad}\rho). \end{aligned}$$

The tangent complexes of the derived rings can be computed via (10.1); here we get:

$$(11.9) \quad \begin{aligned} \mathfrak{t}^0 \mathcal{R}_n &= \ker(H^1(\mathbb{Z}[\frac{1}{S'}], \mathrm{Ad}\rho) \rightarrow H^1(\mathbb{Q}_p, \mathrm{Ad}\rho)/H_f^1), \\ \mathfrak{t}^1 \mathcal{R}_n &\xrightarrow{\sim} H^2(\mathbb{Z}[\frac{1}{S'}], \mathrm{Ad}\rho). \end{aligned}$$

Note that (10.1) actually says that \mathfrak{t}^1 is the kernel of $H^2(\mathbb{Z}[\frac{1}{S'}], \mathrm{Ad}\rho) \rightarrow H^2(\mathbb{Q}_p, \mathrm{Ad}\rho)$, augmented by the cokernel of A , but $H^2(\mathbb{Q}_p, \mathrm{Ad}\rho)$ vanishes by assumption (§10) and the cokernel of A is zero by (11.8).

From (11.9), (11.8) and the assumed vanishing (§10) of $H^2(\mathbb{Q}_v, \mathrm{Ad}\rho)$ for $v \in T$ and for $v = p$, we see that the map

$$(11.10) \quad \mathfrak{t}^1 \mathcal{R}_n \rightarrow \mathfrak{t}^1 \mathcal{S}_n \cong \prod_{v \in Q} H^2(\mathbb{Q}_v, \mathrm{Ad}\rho)$$

is an isomorphism. (As concerns the latter equality: $\mathfrak{t}^1 \mathcal{S}_n$ is a priori a product of Galois cohomology groups with coefficients in the Lie algebra of the torus, but then we can apply Lemma 8.3).

Also the maps

$$(11.11) \quad \mathfrak{t}^i \mathcal{R}_n \rightarrow \mathfrak{t}^i \overline{\mathcal{R}}_n, \mathfrak{t}^i \mathcal{S}_n \rightarrow \mathfrak{t}^i \overline{\mathcal{S}}_n, \mathfrak{t}^i \mathcal{S}_n^{\text{ur}} \rightarrow \mathfrak{t}^i \overline{\mathcal{S}}_n^{\text{ur}}$$

are isomorphisms for $i = 0$ by assumption; and also

$$(11.12) \quad \mathfrak{t}^1 \mathcal{S}_n^{\text{ur}} = 0,$$

since $\mathcal{S}_n^{\text{ur}}$ is formally smooth (see Lemma 8.6, and (8.14)).

We can now verify the claim:

For \mathfrak{t}^0 , we must check that j_1 is an isomorphism in the following diagram, where the rows are the long exact sequences arising from Lemma 4.30 (iv):

$$(11.13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{t}^0(\overline{\mathcal{R}}_n \otimes_{\overline{\mathcal{S}}_n} \overline{\mathcal{S}}_n^{\text{ur}}) & \longrightarrow & \mathfrak{t}^0(\overline{\mathcal{R}}_n) \oplus \mathfrak{t}^0(\overline{\mathcal{S}}_n^{\text{ur}}) & \longrightarrow & \mathfrak{t}^0(\overline{\mathcal{S}}_n) \\ & & \downarrow j_1 & & \downarrow f & & \downarrow g \\ 0 & \longrightarrow & \mathfrak{t}^0(\mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}}) & \xrightarrow{\alpha} & \mathfrak{t}^0(\mathcal{R}_n) \oplus \mathfrak{t}^0(\mathcal{S}_n^{\text{ur}}) & \longrightarrow & \mathfrak{t}^0(\mathcal{S}_n) \end{array}$$

Indeed, by (11.11), f, g are both isomorphisms. Since f is injective, j_1 is injective. Since α is injective, f is surjective and g is injective, we get that j_1 is surjective. Therefore j_1 is an isomorphism, as desired.

For \mathfrak{t}^1 we must check that j_2 is surjective in the following diagram, which is just the continuation of the previous one:

$$(11.14) \quad \begin{array}{ccccccccc} \mathfrak{t}^0(\overline{\mathcal{R}}_n) \oplus \mathfrak{t}^0(\overline{\mathcal{S}}_n) & \longrightarrow & \mathfrak{t}^0(\overline{\mathcal{S}}_n) & \longrightarrow & \mathfrak{t}^1(\overline{\mathcal{R}}_n \otimes_{\overline{\mathcal{S}}_n} \overline{\mathcal{S}}_n) & \longrightarrow & \mathfrak{t}^1(\overline{\mathcal{R}}_n) \oplus \mathfrak{t}^1(\overline{\mathcal{S}}_n) \\ \downarrow f & & \downarrow g & & \downarrow j_2 & & \downarrow \\ \mathfrak{t}^0(\mathcal{R}_n) \oplus \mathfrak{t}^0(\mathcal{S}_n^{\text{ur}}) & \longrightarrow & \mathfrak{t}^0(\mathcal{S}_n) & \xrightarrow{\beta} & \mathfrak{t}^1(\mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}}) & \xrightarrow{\gamma} & \mathfrak{t}^1(\mathcal{R}_n) \oplus \mathfrak{t}^1(\mathcal{S}_n^{\text{ur}}) \xrightarrow{h} \mathfrak{t}^1(\mathcal{S}_n) \end{array}$$

Because of (11.10) and (11.12) the kernel of h is zero. So γ is zero. So β is surjective. We saw above that g is an isomorphism. So j_2 is surjective. \square

12. A PATCHING THEOREM

The current section proves a “patching” theorem for the derived deformation ring. In terms of the discussion of the introduction, we describe the compactness argument required to extract a limit in the context of (1.11).

Theorem 12.1. *Suppose given \mathcal{R}_0 a pro-object of Art_k such that $\mathfrak{t}^i \mathcal{R}_0$ is supported in degrees 0 and 1. Suppose also given a continuous map of (usual) complete local rings $\iota : \mathcal{R}_\infty^\circ \rightarrow \mathcal{R}_\infty$ where $\mathcal{R}_\infty^\circ = W(k)[[X_1, \dots, X_s]]$, $\mathcal{R}_\infty = W(k)[[X_1, \dots, X_{s-\delta}]]$, the map ι makes \mathcal{R}_∞ a finite \mathcal{R}_∞° -module, and*

$$(12.1) \quad \dim \mathfrak{t}^0 \mathcal{R}_0 - \dim \mathfrak{t}^1 \mathcal{R}_0 = \dim(\mathcal{R}_\infty) - \dim(\mathcal{R}_\infty^\circ) (= \delta).$$

Let \mathfrak{a}_n be the descending sequence of ideals of \mathcal{R}_∞° defined as $\mathfrak{a}_n = (p^n, (1 + X_i)^{p^n} - 1)$.

Regard the Artinian rings $R_\infty/\mathfrak{a}_n, S_\infty^\circ/\mathfrak{a}_n, W_n$ as constant objects of pro-Art_k , indexed by a category J that is independent of n . Set

$$\mathcal{C}_n := R_\infty/\mathfrak{a}_n \otimes_{S_\infty^\circ/\mathfrak{a}_n} W_n$$

where \otimes is derived tensor product, as in Definition 3.3, and where the map $S_\infty^\circ \rightarrow W_n$ is the natural augmentation. By functoriality we obtain $e_{n,m} : \mathcal{C}_n \rightarrow \mathcal{C}_m$ for $n > m$.

Finally, suppose given a collection of maps in pro-Art_k

$$(12.2) \quad f_n : \mathcal{R}_0 \rightarrow \mathcal{C}_n,$$

such that, for every $n > m$, if we write $f_{n,m} = e_{n,m} \circ f$ for the composite

$$f_{n,m} : \mathcal{R}_0 \rightarrow \mathcal{C}_n \xrightarrow{e_{n,m}} \mathcal{C}_m$$

we have

$$(12.3) \quad \mathfrak{t}^0 f_{n,m} \text{ is an isomorphism and } \mathfrak{t}^1 f_{n,m} \text{ is a surjection.}$$

Then there is an isomorphism of graded rings

$$\pi_* \mathcal{R}_0 \cong \text{Tor}_{S_\infty^\circ}^*(R_\infty, W(k)),$$

where we understand $\pi_i \mathcal{R}_0$ of the pro-object $(\mathcal{R}_{0,n})$ to be defined as $\varprojlim \pi_i \mathcal{R}_{0,n}$ (for discussion of this definition, see around (3.5)).

Note we don't assume any type of compatibility between the f_n . What we will do instead is to extract a compatible sequence by compactness. Note also that if the map $S_\infty^\circ \rightarrow R_\infty$ is surjective, the Tor-algebra above is actually an exterior algebra on $s - r$ generators.

12.1. The proof of Theorem 12.1. In this proof we shall use a simplicial enrichment of the category pro-Art_k , defined as $\text{pro-Art}_k(A, B) = \lim_i \text{colim}_j \text{Art}_k(A_j, B_i)$, if $A = (j \mapsto A_j)$ and $B = (i \mapsto B_i)$.

Note that this has a nice meaning in terms of functors represented: the q -simplices of $\text{pro-Art}_k(A, B)$ are natural transformations from $\Delta^q \times \text{colim}_i \text{Art}_k(A_i, -)$ to $\text{colim}_j \text{Art}_k(B_j, -)$.

If each A_i is cofibrant and B is *nice*, then the natural map

$$(12.4) \quad \text{holim}_i \text{colim}_j \text{Art}_k(A_j, B_i) \rightarrow \text{pro-Art}_k(A, B)$$

is a weak equivalence, by the argument of Lemma 2.24; we shall only ever consider the simplicial set $\text{pro-Art}_k(A, B)$ when (12.4) holds.

Let us also note that if $\pi_* \mathfrak{t}A$ is finite dimensional, then the space $\text{pro-Art}_k(A, B_i)$ has finite homotopy groups for all i : this reduces to the case $B_i = k \oplus k[n]$ by applying Lemma 2.8. Hence in that case we have (assuming (12.4) as always, and using Lemma A.9):

$$\pi_0(\text{pro-Art}_k(A, B)) = \lim_i \pi_0(\text{pro-Art}_k(A, B_i)).$$

We shall write $[A, B]$ for this profinite set. The above equation then implies $[A, B] = \lim_i [A, B_i]$.

We wish to apply the above discussion for $A = \mathcal{R}_0$ and $B = \mathcal{C}_n$, and see that $[\mathcal{R}_0, \mathcal{C}_n]$ is in fact a finite set. By definition, the levels of the pro-object $B = \mathcal{C}_n$ have the homotopy types $B_i \simeq \tau_{\leq i}(\mathcal{R}_\infty/\mathfrak{a}_n \otimes_{S_\infty^\circ/\mathfrak{a}_n} W_n^c)$, where W_n^c is a cofibrant replacement of W_n as an algebra over $S_\infty^\circ/\mathfrak{a}_n$. Up to homotopy B_{i+1} may be obtained from B_i by taking homotopy fibers of a map to $k \oplus k[i+2]$ finitely many times (see Lemma 2.8 again), so the assumption that $\pi_* \mathfrak{t}\mathcal{R}_0$ vanishes in degrees besides 0, -1 implies that the map $[A, B_{i+1}] \rightarrow [A, B_i]$ is bijective for $i \geq 1$. Just as above, $[A, B_i]$ is finite for all i . Hence $[A, B]$ is finite in this case.

For any $[f] \in [\mathcal{R}_0, \mathcal{C}_n]$ there are well defined induced maps

$$\mathfrak{t}^i(e_{n,m} \circ f) : \mathfrak{t}^i \mathcal{C}_m \rightarrow \mathfrak{t}^i \mathcal{C}_n \rightarrow \mathfrak{t}^i \mathcal{R}_0$$

for all $m \leq n$ and all i . As n varies the finite sets $[\mathcal{R}_0, \mathcal{C}_n]$ form an inverse system, and we shall consider the subsystem

$$X_n = \{[f] \in [\mathcal{R}_0, \mathcal{C}_n] \mid \forall m \leq n : \mathfrak{t}^i(e_{n,m} \circ f) \text{ is iso for } i = 0 \text{ \& epi for } i = 1\}.$$

By assumption $X_n \neq \emptyset$ for all $n \geq 0$. Hence this is an inverse system of non-empty finite sets, so by compactness the inverse limit is non-empty.

We may therefore pick morphisms $g_n : \mathcal{R}_0 \rightarrow \mathcal{C}_n$ representing compatible elements of the subset $X_n \subset [\mathcal{R}_0, \mathcal{C}_n]$ and hence simplicial homotopies between g_n and the composition $e_{n+1,n} \circ g_{n+1}$; that is to say, there exists a 1-simplex in the mapping space $\text{pro-Art}_k(\mathcal{R}_0, \mathcal{C}_n)$ with these vertices. This 1-simplex induces a natural simplicial homotopy between the natural transformations of functors

$$\text{Hom}(\mathcal{C}_n, -) \rightarrow \text{Hom}(\mathcal{R}_0, -).$$

induced by g_n and by $e_{n+1,n} \circ g_{n+1}$. This data induces a natural transformation of functors $\text{Art}_k \rightarrow \text{sSets}$

$$\text{hocolim}_n \text{Hom}(\mathcal{C}_n, -) \rightarrow \text{Hom}(\mathcal{R}_0, -)$$

which is then also an isomorphism in \mathfrak{t}^0 and an epimorphism on \mathfrak{t}^1 .

We claim that it is in fact an isomorphism on \mathfrak{t}^i for all i , for which it suffices to verify that both sides vanish for $i \notin \{1, 0\}$, and that both sides have the same ‘‘Euler characteristic.’’

The vanishing is true for $\mathfrak{t}^* \mathcal{R}_0$ by assumption. As for $\varinjlim_n \mathfrak{t}^i \mathcal{C}_n$, there’s an exact triangle in the derived category of k -modules

$$\mathfrak{t} \mathcal{C}_n \rightarrow \mathfrak{t}(\mathcal{R}_\infty/\mathfrak{a}_n) \oplus \mathfrak{t}(W_n) \rightarrow \mathfrak{t}(S_\infty^\circ/\mathfrak{a}_n) \xrightarrow{[1]}$$

and taking cohomology and taking direct limit as $n \rightarrow \infty$, we get

$$\mathfrak{t}^i \mathcal{C}_n \rightarrow \mathfrak{t}^i \mathcal{R}_\infty \oplus \underbrace{\mathfrak{t}^i W(k)}_0 \rightarrow \mathfrak{t}^i S_\infty^\circ \xrightarrow{[1]}.$$

Here we have used $\mathfrak{t}^i \mathcal{R}_\infty = \varinjlim \mathfrak{t}^i(\mathcal{R}_\infty/\mathfrak{a}_n)$ and similarly for S_∞° , because \mathcal{R}_∞ considered in pro-Art_k can be defined by the projective system $\mathcal{R}_\infty/\mathfrak{a}_n$ inside Art_k .⁹

⁹Details: we need to see that \mathfrak{a}_n defines the standard profinite topology on \mathcal{R}_∞ . Write \mathfrak{m}_R for the maximal ideal of \mathcal{R}_∞ . On the one hand, the rings $\mathcal{R}_\infty/\mathfrak{a}_n$ are Artin because \mathcal{R}_∞ is module-finite

Now the claimed vanishing follows from our computation of the tangent complex for power series rings (7.18), and the Euler characteristic claim follows from our assumption (12.1).

Hence the pro-objects \mathcal{R}_0 and $(n \mapsto \mathcal{C}_n)$ represent equivalent functors, and hence the induced map of homotopy groups

$$\varprojlim \pi_* \mathcal{R}_0 \rightarrow \varprojlim \pi_* \mathcal{C}_n$$

is also (see (3.4)) an isomorphism for all i . This concludes the proof of Theorem 12.1 since

$$\varprojlim \pi_i \mathcal{C}_n = \varprojlim \mathrm{Tor}_{S_\infty^\circ/\mathfrak{a}_n}^i(R_\infty/\mathfrak{a}_n, W_n) \cong \mathrm{Tor}_{S_\infty^\circ}^i(R_\infty, W).$$

(for the last step see Lemma 7.6; for the computation of homotopy groups of a tensor product, see [28, Theorem 6]).

13. BACKGROUND ON THE OBSTRUCTED TAYLOR–WILES METHOD, AFTER CALEGARI–GERAGHTY

We present the “obstructed” Taylor–Wiles method, in the form given by Khare and Thorne [20]; it is originally discovered by Calegari–Geraghty [6], and another version has been developed by D. Hansen [15]. This section has no original ideas (although any errors are due to us). It is largely independent of the rest of the paper; it just provides the input for us to apply our previous theorems. It uses the conjecture on the existence of Galois representations, as formulated in Conjecture 6.1, and outputs a set of Taylor–Wiles primes Q_n where one has very good control, if not on all of $\pi_0 \mathcal{R}_n$, then at least on a “very large” Artinian quotient of it.

13.1. Notation and setup. Let us try to describe more carefully our setup. Thus far we have dealt with an abstract Galois representation ρ . We will henceforth be considering the case where ρ comes from a modular form, and we will specify some details about this modular form.

- (1) As before, S will be a finite set of primes containing p , and $T = S - \{p\}$.
- (2) As before (§6.5) \mathbf{G} is a split semisimple \mathbb{Q} -group dual to G . We suppose that \mathbf{G} admits a smooth reductive model over $\mathbb{Z}[\frac{1}{T}]$.
- (3) We let Y_0 be the arithmetic manifold associated to a subgroup $K_0 = \prod K_{0,v}$, i.e.

$$(13.1) \quad Y_0 = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_\infty^\circ K_0,$$

as in §6.5.

Here we suppose that $K_{0,v}$ is obtained from the integral points of \mathbf{G} for $v \notin T$, and for $v \in T$ we take $K_{0,v}$ to be an Iwahori subgroup. (Informally, this means one takes the preimage of a Borel subgroup over the residue field. For a formal definition see [35, §3.7]).

over S_∞° by assumption; therefore there exists n_1 such that $\mathfrak{m}_R^{n_1} \subset \mathfrak{a}_n$. On the other hand, $\mathfrak{a}_n \subset \mathfrak{m}_R^n$: clearly $p^n \in \mathfrak{m}_R^n$, and if $Y \in \mathfrak{m}_R$, then $(1 + Y)^{p^n} - 1 = \sum_{j \geq 1} \binom{p^n}{j} Y^j$ belongs to \mathfrak{m}_R^n too.

More generally, we define $Y(K)$ just as in (13.1) for an open compact subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$; we will only deal with examples where K decomposes as a product $\prod_v K_v$.

- (4) We follow [20] on Hecke algebras:

Consider the chain complex of Y_0 as an object in the derived category of \mathbb{Z} -modules; each Hecke operator gives an endomorphism of this object. The Hecke algebra T_0 for Y_0 then means the ring of endomorphisms generated by all Hecke operators prime to the level. (This has a few minor advantages, in particular, T_0 acts on cohomology with any coefficients.)

For some larger level K' , the Hecke algebra $T_{K'}$ is defined in the same way (again, Hecke operators relatively prime to the level).

Warning: we will use at certain points slight variations of this definition, in particular enlarging the Hecke operator by using certain operators at primes dividing the level, but if used without explanation, “Hecke algebra” should be taken in the sense just defined.

- (5) The invariants q, δ :

It is known (see [4, III, §5.1]), [4, VII, Theorem 6.1] and for the non-compact case [5, 5.5]) that the tempered cuspidal cohomology

$$H^*(Y(K), \mathbb{C})_{\text{temp}}$$

of $Y(K)$, i.e. that part of the cohomology associated to tempered cuspidal representations under the standard indexing of cohomology by representations, is concentrated in degrees $[q, q + \delta]$. Here $2q + \delta = \dim Y(K)$ and δ is the difference $\text{rank } \mathbf{G}(\mathbb{R}) - \text{rank } K_\infty$.

- (6) Fix a Hecke eigenclass $f \in H^q(Y_0, \overline{\mathbb{Q}})_{\text{temp}} := H^q(Y_0, \overline{\mathbb{Q}}) \cap H^q(Y(K), \mathbb{C})_{\text{temp}}$; let K_f be the field generated by all Hecke eigenvalues, and let \mathcal{O} be the ring of integers of K_f . Thus f defines a homomorphism

$$T_0 \longrightarrow \mathcal{O}$$

from the Hecke algebra for Y_0 to \mathcal{O} .

- (7) Let \wp be a prime of \mathcal{O} , above the rational prime p , such that
- (a) $H^*(Y_0, \mathbb{Z})$ is p -torsion free.
 - (b) p is “large” relative to \mathbf{G} : larger than the order of the Weyl group.
 - (c) “No congruences between f and other forms:” Writing \mathfrak{m} for the maximal ideal of T_0 given by the kernel of $T_0 \rightarrow \mathcal{O}/\wp$, the induced map of completions $(T_0)_{\mathfrak{m}} \rightarrow \mathcal{O}_{\wp}$ is an isomorphism.
 - (d) \mathcal{O} over \mathbb{Z} is unramified at \wp .
 - (e) The localization $H_j(Y_0, \mathbb{Z}_p)_{\mathfrak{m}}$ vanishes in degrees $j \notin [q, q + \delta]$.

One expects that all the conditions are satisfied for all but finitely many \wp .

Note also that if $K' \supset K_0$ is some deeper level structure, with Hecke algebra $T_{K'}$, there is still a morphism $T_{K'} \rightarrow \mathcal{O}$; as an abuse of notation we will still use \mathfrak{m} for the correspondingly defined maximal ideal, and $T_{K', \mathfrak{m}}$ the completion of $T_{K'}$.

(8) Set $k = \mathcal{O}/\wp$, the residue field of \wp . Let

$$\rho : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(k)$$

be the Galois representation (modulo \wp) associated to f (see Conjecture 6.1). Again, by Conjecture 6.1, it is the reduction of a representation

$$(13.2) \quad \rho_{\mathcal{O}} : \pi_1 \mathbb{Z}[\frac{1}{S}] \rightarrow G(\mathcal{O}_{\wp} = W(k)).$$

(9) We suppose that the image of ρ satisfies the conditions of §10, i.e. (among others) big image, trivial deformation theory at T , and crystalline with small weights at p .

We prove the following result (we apologize that to precisely formulate the Theorem here, we have to make a few forward references to later in the text):

Theorem 13.1. *Let assumptions be as in §13.1 and assume also Conjecture 6.1. Let q be a large enough integer, and let $s = q \cdot \text{rank}(G)$. Let δ be the “defect” of G as in (5) of §13.1. Set*

$$S_{\infty}^{\circ} = W(k)[[X_1, \dots, X_s]], R_{\infty} = W(k)[[X_1, \dots, X_{s-\delta}]],$$

Let \mathfrak{a}_n be the ideal of S_{∞}° defined by

$$(13.3) \quad \mathfrak{a}_n = (p^n, (1 + X_i)^{p^n} - 1)$$

inside S_{∞}° ; thus $S_{\infty}^{\circ}/\mathfrak{a}_n$ is naturally identified with the \mathbb{Z}/p^n -group algebra of $(\mathbb{Z}/p^n)^s$. Then we can find the following data:

- (a) Homomorphisms $S_{\infty}^{\circ} \xrightarrow{\iota} R_{\infty} \twoheadrightarrow \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ of complete local rings, whose composite is the natural augmentation $S_{\infty}^{\circ} \rightarrow W \rightarrow \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$.
- (a)' For each integer n , allowable Taylor–Wiles data Q_n of level n , with associated covering groups Δ_n ((13.8)), group rings $\overline{S}_n^{\circ} := W_n[\Delta_n]$, and deformation rings \mathcal{R}_n ;
- (b)' An explicit function $K(n) \rightarrow \infty$ and for each integer n , homomorphisms f_n, g_n rendering commutative the diagram

$$(13.4) \quad \begin{array}{ccccc} S_{\infty}^{\circ} & \xrightarrow{\iota} & R_{\infty} & \longrightarrow & \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \\ \downarrow f_n & & \downarrow g_n & & \downarrow \\ \overline{S}_n^{\circ} & \longrightarrow & \overline{R}_n & \longrightarrow & \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} / (p^n, \mathfrak{m}^{K(n)}) \end{array}$$

where:

- we write \overline{R}_n for the quotient of $\pi_0 \mathcal{R}_n / (p^n, \mathfrak{m}^{K(n)})$ that classifies deformations of inertial level $\leq n$ (see (8.10)) at all primes in Q_n ,
- the map $\overline{S}_n^{\circ} = W_n[\Delta_n] \rightarrow \overline{R}_n$ is the natural one (explained in and before (13.16)).
- All the maps commute with the natural augmentations to k , i.e. are local homomorphisms.

Also $f_n(\mathfrak{a}_n) = 0$, and the induced maps

$$(13.5) \quad \begin{cases} S_\infty^\circ/\mathfrak{a}_n \longrightarrow \overline{S}_n^\circ, \\ R_\infty/\mathfrak{a}_n \cong R_\infty \otimes_{S_\infty^\circ} \overline{S}_n^\circ \longrightarrow \overline{R}_n \end{cases}$$

are both isomorphisms.

- (d) A complex D_∞ of finite free S_∞° -modules, equipped with a compatible R_∞ -action in the derived category of S_∞° -modules; and equipped with an isomorphism

$$H_*(D_\infty \otimes_{S_\infty^\circ} W) \cong H_*(Y_0, W)_\mathfrak{m},$$

which is compatible for the R_∞ -actions. Here R_∞ acts on the right-hand side via the map $R_\infty \rightarrow \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ from (a), followed by the action of $\pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ supplied by Conjecture 6.1. Moreover, $H_*(D_\infty)$ is concentrated in degree q and $H_q(D_\infty)$ is a finite free R_∞ -module.

- (e) [Easy consequence of (d):] The natural surjections

$$(13.6) \quad R_\infty \otimes_{S_\infty^\circ} W \twoheadrightarrow \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \rightarrow \text{image of Hecke algebra on } H_q(Y_0, W)_\mathfrak{m}$$

are both isomorphisms. The action of $R_\infty \otimes_{S_\infty^\circ} W$ on $H_*(Y_0, W)_\mathfrak{m}$ extends to a free graded action of $\text{Tor}_*^{S_\infty^\circ}(R_\infty, W)$ on $H_*(Y_0, W)_\mathfrak{m}$, over which this homology is freely generated in degree q .

This is exactly the setup needed to apply the results of the prior sections. For later use, observe that our assumptions (7(e) of §13.1) imply that the “image of the Hecke algebra” appearing in (13.6) is simply W ; then (e) implies that $S_\infty^\circ \rightarrow R_\infty$ is surjective.

13.2. Outline of the argument. We briefly outline how the argument goes. The discussion is somewhat vague, but we hope it is better than saying nothing at all.

The sets Q_n will be allowable Taylor–Wiles data (§6.7). The cohomological condition in “allowable” means that we know exactly the minimal number of generators for

$$R_n := \pi_0 \mathcal{R}_n$$

(the usual deformation ring at level $S \coprod Q_n$); this number is $s - \delta$. Any choice of elements $x_1, \dots, x_{s-\delta} \in \mathfrak{m}(R_n)$ projecting to a basis for $\mathfrak{m}/\mathfrak{m}^2$ results in surjections

$$W[[x_1, \dots, x_{s-\delta}]] \twoheadrightarrow R_n \twoheadrightarrow k[[x_1, \dots, x_{s-\delta}]]/(x_i x_j).$$

Informally speaking, the content of the Theorem is that R_n actually becomes closer and closer to $W[[x_1, \dots, x_{s-\delta}]]$ as n gets large. To prove this, we construct a morphism

$$(13.7) \quad S_\infty^\circ \rightarrow R_n$$

which comes, in the end, from studying the action of the inertia group at places in Q_n (see (13.10)). We try to show that the image of S_∞° is “big”. The map ι of Theorem 13.1 also arises by taking a “limit” of the maps (13.7), using a compactness argument.

To prove that the image of S_∞° is big, we study the action of R_n on a space of modular forms – indeed, the homology of a certain arithmetic manifold obtained by adding level Q_n . We claim the pullback of this action to S_∞° has kernel that is “not too large.” The action of R_n comes from Conjecture 6.1, and the “local-to-global compatibility”, which we shall state more precisely in Section 13.5, asserts that the resulting action of S_∞° agrees with the action defined geometrically via “diamond operators,” i.e. out of the action of explicit groups Δ_n of automorphisms of the underlying arithmetic manifold. (Studying such genuine automorphisms is much easier than studying Hecke operators.)

The key point here comes from the fact that Δ_n acts freely on the underlying arithmetic manifold, and so the homology is computed from a complex of free modules whose length is tightly bounded from above. Then the Auslander-Buchsbaum theorem states, roughly, that if a module has a short projective resolution, then its annihilator cannot be too large. The relevance of this last point is a key observation of Calegari and Geraghty.

If the reader will forgive one more vagueness, the mechanism of this last argument is similar to a well-known mechanism that prohibits free group actions of a finite group Δ on a homology sphere M : the quotient M/Δ has homology “approximated” by that of Δ , and thus has difficulty looking like the homology of a manifold.

13.3. Construction of complexes. A key point for the analysis is to construct complexes which compute the cohomology of various $Y(K)$ ’s localized at \mathfrak{m} . These complexes should be of reasonable size (much smaller than the full chain complex) so that one can make compactness arguments. These should also carry actions of the usual deformation ring, when we consider them as objects in the derived category. We summarize briefly here the properties of these complexes and how they are constructed.

Lemma 13.2. *Assume Conjecture 6.1. Then for each complete local W -algebra E and each pair $(K \triangleleft K')$ as in §6.6, with $\Delta = K'/K$ acting in the natural way on $Y(K)$, we may construct an object $C_*^\Delta(Y(K); E)_\mathfrak{m}$ in the derived category of $E\Delta$ -modules¹⁰, equipped with a Δ -equivariant identification of its homology with $H_*(Y(K); E)_\mathfrak{m}$, in such a way that the following properties are satisfied:*

- a. *The object $C_*^\Delta(Y(K), E)_\mathfrak{m}$ is quasi-isomorphic to a bounded complex of finite free $E\Delta$ -modules. (In fact, we will just fix such a quasi-isomorphism and use $C_*^\Delta(Y(K), E)_\mathfrak{m}$ to denote that complex.)*
- b. *Let R_K be the (usual, underived) deformation ring classifying deformations of ρ that are unramified at each place where K is hyperspecial. There is an action of R_K on $C_*^\Delta(Y(K); E)_\mathfrak{m}$ by endomorphisms in the derived category of $E\Delta$ -modules, satisfying, at the level of homology, the usual compatibility between Frobenius and Hecke operators.*

¹⁰Despite the notation, $C_*^\Delta(Y(K); E)_\mathfrak{m}$ is not intended as the localization or completion of some complex at \mathfrak{m} !

c. We have descent from level K to level K' , i.e. there is an isomorphism

$$C_*^\Delta(Y(K), E)_\mathfrak{m} \otimes_{E\Delta} E \simeq C_*(Y(K'), E)_\mathfrak{m}$$

where $\otimes_{E\Delta} E$ refers to the derived tensor product, a functor from the derived category for $E\Delta$ to that for E . This homomorphism is compatible with the homomorphism $R_K \rightarrow R_{K'}$.

Proof. (Outline, summarizing the work of Khare–Thorne [20]):

The main point is that we can do the localization at \mathfrak{m} at the derived level:

Take first of all $C_*^\Delta(Y(K), E)$ to be the chain complex of $Y(K)$ with E coefficients, considered as an object in the derived category of $E\Delta$ -modules. Let T_K be the Hecke algebra at level K . (For the current discussion this means the Hecke algebra generated by prime-to- K operators, as in §13.1; however, the same arguments will apply in the setting of §13.5, where we will use a slightly larger Hecke algebra.)

As before we define $T_{K,\mathfrak{m}}$ as the completion of T_K at \mathfrak{m} ; it is a direct factor of the p -completion of T_K . Denote by e the corresponding idempotent; it is an endomorphism of $C_*^\Delta(Y(K), E)$ in the derived category, and thus [3, Proposition 3.4] there is a splitting $C_* = C_*e \oplus C_*e'$ corresponding to $1 = e + e'$; the resulting splitting is unique up to unique isomorphism (“unique isomorphism” understood to be in the derived category). Thus we put

$$C_*^\Delta(Y(K), E)_\mathfrak{m} := C_*^\Delta(Y(K), E) \cdot e.$$

Its homology is finitely generated as $E\Delta$ -module and we obtain (a) by passing to a minimal resolution. Part (b) follows at once from the Conjecture enunciated above.

For descent, part (c), we need to be careful since the Hecke algebras acting at different levels are not quite the same; for discussion of that issue in the case needed we refer to [20, Lemma 6.6]. \square

13.4. Taylor–Wiles sets. Fix an allowable Taylor–Wiles datum Q of level n (see §6.7); recall this comes equipped with a choice of element of $T(k)$ conjugate to Frobenius, for each prime in Q .

If Q is an auxiliary set of primes, disjoint from S , we let $Y_0(Q)$ be the manifold obtained by adding Iwahori level at Q , i.e. replacing $K_0 = \prod K_{0,v}$ by

$$\prod_{v \notin Q} K_{0,v} \times \prod_{v \in Q} \text{Iwahori}_v.$$

As usual (generalizing the case of the covering of modular curves $X_1(q) \rightarrow X_0(q)$) we can produce a manifold $Y_1(Q)$ which covers $Y_0(Q)$ with Galois group

$$\prod_{q \in Q} \mathbf{T}(\mathbb{F}_q)$$

where \mathbf{T} was, we recall, a maximal split torus in \mathbf{G} .

From the surjection $\mathbf{T}(\mathbb{F}_q) \twoheadrightarrow \mathbf{T}(\mathbb{F}_q)/p^n \cong (\mathbb{Z}/p^n)^r$ (where r is the rank of G) we get a unique subcovering $Y(Q, n)$ with Galois group

$$(13.8) \quad \Delta_Q := \prod_{q \in Q} \mathbf{T}(\mathbb{F}_q)/p^n \cong (\mathbb{Z}/p^n)^{r \cdot \#Q}$$

over $Y_0(Q)$. In other words, this fits in the following diagram

$$\begin{array}{c} \prod_{q \in Q} \mathbf{T}(\mathbb{F}_q) \\ \hline Y_1(Q) \rightarrow Y(Q, n) \rightarrow Y_0(Q) \\ \hline \Delta_Q \cong (\mathbb{Z}/p^n)^{r \cdot \#Q} \end{array}$$

Δ_Q also enters into the deformation theory of Galois representations at level Q , as we now recall:

Let $R_S \amalg Q$ be the usual deformation ring for Galois representations at level $S \amalg Q$, which are, as always, crystalline at p . Thus $R_S \amalg Q = \pi_0 \mathcal{R}_S \amalg Q$ in the notation of §11.2. Form a universal deformation

$$\tilde{\rho}^{\text{univ}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow G(R_S \amalg Q)$$

and restrict this to the Galois group of a place $q \in Q$.

This restricted representation can be conjugated, by Remark 8.4, to take values in T . Note that the resulting conjugated homomorphism $\pi_1(\mathbb{Q}_v) \rightarrow T(R_S \amalg Q)$ is uniquely determined by requiring that the image of Frobenius reduce to the element of $T(k)$ prescribed as part of the Taylor–Wiles data (see first paragraph of this subsection). We shall suppose that this has been done.

We therefore get a map of deformation rings

$$\text{framed deformation ring of } \rho_{\mathbb{Q}_q}^T, \text{ with target group } T \longrightarrow R_S \amalg Q.$$

Now the left-hand side is identified with the group algebra of a certain quotient of $\mathbf{T}(\mathbb{Q}_q)$ (Remark 8.7) and in particular we obtain a map

$$(13.9) \quad \mathbf{T}(\mathbb{F}_q)_p \rightarrow R_S^* \amalg Q.$$

where the subscript p means “ p -part”, i.e. $A_p = A \otimes \mathbb{Z}_p$.

(More explicitly, we can factorize $\tilde{\rho}^{\text{univ}}|_{\pi_1 \mathbb{Q}_q} : (\pi_1 \mathbb{Q}_q)^{\text{ab}} \rightarrow T(R_S \amalg Q)$, and if we compose this with any character $\chi : T \rightarrow \mathbb{G}_m$, we get a homomorphism $(\pi_1 \mathbb{Q}_q)^{\text{ab}} \rightarrow R_S^* \amalg Q$; in particular, restricting to $\mathbb{F}_q^* \subset (\pi_1 \mathbb{Q}_q)^{\text{ab}}$ and noting that the image is pro- p , we get a map $\mathbb{F}_q^* \otimes \mathbb{Z}_p \rightarrow R_S^* \amalg Q$. Thus, we have produced a map $\mathbb{F}_q^* \otimes \mathbb{Z}_p \rightarrow R_S^* \amalg Q$ for each character $T \rightarrow \mathbb{G}_m$ – which, said intrinsically, amounts to (13.9)).

Taking a product over $q \in Q$, we get at last a morphism

$$(13.10) \quad \widetilde{\Delta}_Q := \prod_{q \in Q} \mathbf{T}(\mathbb{F}_q)_p \rightarrow R_S^* \amalg Q$$

Note that $\widetilde{\Delta}_Q/p^n = \Delta_Q$. Observe that $\widetilde{\Delta}_Q$ is isomorphic to $\prod_{q \in Q} (\mathbb{Z}/p^{n'_q})^r$, where n'_q is the highest power of p dividing $q - 1$.

13.5. Enlarging \mathfrak{m} . In the classical theory of modular forms, the space of modular oldforms at level q is a sum of two copies of the space of modular forms at level 1; one can distinguish these two copies as the two eigenspaces for the U_q -operator. We carry out the analogue here, using the “ U_q -operator” to slightly enlarge the maximal ideal \mathfrak{m} .

Let I_q be an Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_q)$ and $I'_q \leq I_q$ the subgroup corresponding to $Y(Q, n)$, so that $I_q/I'_q \simeq (\mathbb{Z}/p^n)$. Writing X_* for the co-character group of \mathbf{T} (more canonically one replaces \mathbf{T} by the torus quotient of a Borel subgroup) and let $X_*^+ \subset X_*$ be the positive submonoid defined by the Borel subgroup \mathbf{B} (the dual to the cone spanned by roots). In the usual way each $\chi \in X_*^+$ gives rise to a Hecke operator at level $Y(Q, n)$, namely the operator defined by the double coset $I'_q \chi(q) I'_q$; for $\mathbf{G} = \mathrm{PGL}_2$ and χ a generator for X_*^+ this gives the U_q -operator.

Let the extended Hecke algebra at level $Y(Q, n)$ be the subalgebra generated¹¹ by prime-to-level Hecke operators together with all the $I'_q \chi(q) I'_q$, as above. We make an ideal \mathfrak{m}' of this extended Hecke algebra thus:

$$\mathfrak{m}' = (\mathfrak{m}, I'_q \chi(q) I'_q - \langle \chi, \mathrm{Frob}_q^T \rangle).$$

Here, we have used the choice of $\mathrm{Frob}_q^T \in T(k)$ that comes with a Taylor–Wiles prime, and the identification $\chi \in X_*(\mathbf{T}) \simeq X^*(T)$ to form $\langle \chi, \mathrm{Frob}_q^T \rangle \in k$.

We can now formulate:

Assumption 2. (Local-global compatibility): *Let Q be a Taylor–Wiles datum of level n , and let \mathfrak{m}' be as defined in the preceding two paragraphs. The action of $\tilde{\Delta}_Q$ on $C_*^{\Delta_Q}(Y(Q, n); E)_{\mathfrak{m}'}$ via (13.10) and Conjecture 6.1 coincides with the natural action, that is to say, the action via $\tilde{\Delta}_Q \rightarrow \Delta_Q$ and deck transformations.*

13.6. Sequences of Taylor–Wiles sets. Now choose arbitrarily allowable Taylor–Wiles data (Q_n, n) , indexed by an increasing sequence of integers n , and write for short $\Delta_n = \Delta_{Q_n}$ and $\tilde{\Delta}_n = \tilde{\Delta}_{Q_n}$ (see (13.10) for definition). We choose all the Q_n to be of the same size $\#Q$; recall that

$$(13.11) \quad \tilde{\Delta}_n/p^n = \Delta_n \cong (\mathbb{Z}/p^n)^{r\#Q}$$

where r is the rank of G .

Let C_n be the reduced chain complex constructed in Assumption 2 for $Y_n = Y(Q_n, n)$ with $E = W_n$ and with maximal ideal \mathfrak{m}' as in §13.5, i.e.

$$(13.12) \quad C_n = C_*^{\Delta_n}(Y(Q_n, n), W_n)_{\mathfrak{m}'},$$

so it computes the homology of $Y(Q_n, n)$ with W_n coefficients, localized at \mathfrak{m}' , and equivariantly for the action of the Galois group Δ_n of $Y(Q_n, n) \rightarrow Y_0(Q)$.

The complex C_n is by definition a complex of \overline{S}_n° -modules, where

$$S_n^\circ := W[\Delta_n], \quad \overline{S}_n^\circ = S_n^\circ/p^n.$$

¹¹As with the Hecke algebra, we regard these as acting on the chain complex of $Y(Q, n)$, considered as an object of a suitable derived category, and “generated” is taken inside the endomorphism ring of that object.

We observe for later use that we have an identification

$$(13.13) \quad C_n \otimes_{\overline{S}_n^\circ} W_n \simeq C(Y_0, W_n)_{\mathfrak{m}},$$

inside the derived category of W_n -modules. In fact, part (c) of Lemma 13.2 says that the left-hand side is identified with $C(Y_0(Q_n), W_n)_{\mathfrak{m}'}$, where \mathfrak{m}' is the analogue to \mathfrak{m}' above but at level $Y_0(Q_n)$; and then the natural projection $Y_0(Q_n) \rightarrow Y_0$ induces an isomorphism $C(Y_0(Q_n), W_n)_{\mathfrak{m}'} \xrightarrow{\sim} C(Y_0, W_n)_{\mathfrak{m}}$ (this corresponds to [20, Lemma 6.25 (4)]). Note that (13.13) implies that

$$(13.14) \quad H_j(C_n) = 0, \quad j \notin [q, q + \delta].$$

because, by 5 and 7 from §13.1, the homology $H_*(Y_0, k)_{\mathfrak{m}}$ is vanishing for $j \notin [q, q + \delta]$. In particular, we can and do suppose that C_n itself is supported in degrees $[q, q + \delta]$.

We let

$$R_n = \text{usual deformation ring at level } S \coprod Q_n.$$

By virtue of our assumptions (Lemma 13.2) there is a natural action of R_n on C_n , by endomorphisms in the derived category of \overline{S}_n° -modules. The assumption of local-global compatibility (Assumption 2) means that this action factors through the quotient

$$(13.15) \quad \theta : R_n \rightarrow \underbrace{R_n \otimes_{W[\tilde{\Delta}_n]} W_n[\Delta_n]}_{\text{inertial level} \leq n}$$

We can think of the right hand side as classifying deformations of inertial level $\leq n$ (8.10) at primes in Q – that is to say, deformations such that, at each prime $q \in Q$, the deformation factors not merely through the $(\mathbb{Z}/q)^*$ quotient of inertia, but actually through the maximal exponent p^n quotient of that group. In particular, we have a natural map

$$(13.16) \quad \Delta_n \rightarrow \left(R_n \otimes_{W[\tilde{\Delta}_n]} W_n[\Delta_n] \right)^*$$

To do the limit process we replace R_n (which might, *a priori*, be a bit too large – e.g., infinite) by an Artinian quotient:

Lemma 13.3. *Suppose that D_n is any complex of finite free \overline{S}_n° -modules. Let \mathbf{e} be an endomorphism of D_n , considered as an object in the derived category, which acts nilpotently on homology. Then $\mathbf{e}^{K(n)} = 0$, where we can take $K(n) = \ell \cdot Q \cdot o$, with*

- ℓ the length of \overline{S}_n° over itself;
- o the range of degrees in which D_n is supported (i.e., if supported in $[a, b]$, we have $o = b - a + 1$);
- Q is the total rank of D_n over \overline{S}_n° .

Without loss of generality we can and do always suppose $K(n) \geq 2n$.

Proof. The total length of homology, as a \overline{S}_n° -module, is clearly at most $Q \cdot \ell$. This shows that $e^{Q\ell}$ acts trivially on homology. Using [20, Lemma 2.5] the claimed result follows. \square

With $K(n)$ as in the above lemma, applied to C_n in place of D_n , put

$$\mathfrak{d}_n = \left(\mathfrak{m}_{R_n}^{K(n)}, \ker(\theta) \right) \subset R_n$$

where θ is as in (13.15). Then

$$\overline{R}_n = R_n / \mathfrak{d}_n$$

is Artinian, and the action of R_n on C_n factors through \overline{R}_n . The map from $\tilde{\Delta}_n$ to R_n yields a map from the group algebra of Δ_n into \overline{R}_n , i.e.

$$(13.17) \quad \overline{S}_n^\circ \rightarrow \overline{R}_n$$

and we also have descent

$$(13.18) \quad \overline{R}_n \otimes_{\overline{S}_n^\circ} W_n \cong \pi_0 \mathcal{R}_S / (p^n, \mathfrak{m}^{K(n)})$$

which expresses the fact that “a Galois representation at level Q_n , which is trivial on Δ_n , actually descends to base level,” i.e. the underived version of the discussion of §8.¹²

Finally, the assumption of local-global compatibility from the previous sections still says that the action of S_n° on C_n via $\overline{S}_n^\circ \rightarrow \overline{R}_n$ (see (13.17)) coincides with the natural action of \overline{S}_n° on C_n (i.e. the natural action by deck transformations.)

13.7. Numerology. Recall (6.8) that, the Q_n being an allowable Taylor–Wiles datum of level n , we have with $S' = S \coprod Q_n$, a surjection

$$A : H^1(\mathbb{Z}[\frac{1}{S'}], \text{Ad}\rho) \twoheadrightarrow H^1(\mathbb{Q}_p, \text{Ad}\rho) / H_f^1(\mathbb{Q}_p, \text{Ad}\rho).$$

The kernel of A describes the minimum number of generators for R_n as a $W(k)$ -module. An Euler characteristic computation shows that $\ker(A)$ has dimension

$$\underbrace{\dim(B) - \delta}_{\dim H^1(\mathbb{Z}[\frac{1}{S}], \text{Ad}\rho) - \dim H^2(\mathbb{Z}[\frac{1}{S}], \text{Ad}\rho)} - \underbrace{\dim G}_{\dim H^1(\mathbb{Q}_p, \text{Ad}\rho)} + \underbrace{\dim U}_{\dim H_f^1} + \underbrace{(\text{rank } G) \cdot \#Q}_{\oplus_Q H^2}$$

which equals $r \cdot \#Q - \delta$; recall that r is the rank of G . In other words, R_n is a quotient of a power series ring over $W(k)$ in $r\#Q - \delta$ variables. We set

$$s = r \cdot \#Q.$$

and will choose $\#Q$ so large that $\pi_0 \mathcal{R}_S$ can be generated over $W(k)$ by $s - \delta$ elements.

¹²More precisely, $\overline{R}_n \otimes_{\overline{S}_n^\circ} W_n$ computes the quotient of R_n by the ideal generated by elements $\delta - 1$ with $\delta \in \tilde{\Delta}_n$, by p^n , and by $\mathfrak{m}^{K(n)}$. The quotient of R_n by the ideal generated by $\delta - 1$ recovers $\pi_0 \mathcal{R}_S$, whence (13.18).

13.8. Limit rings. Now we pass to the limit in the following way. As we just proved in §13.7, each R_n and so also \bar{R}_n is a quotient of $R_\infty := W(k)[[x_1, \dots, x_{s-\delta}]]$. Choose once and for all a surjection $R_\infty \rightarrow \pi_0 \mathcal{R}_S$ and choose $R_\infty \rightarrow \bar{R}_n$ to lift the resulting map $R_\infty \rightarrow \pi_0 \mathcal{R}_S/(p^n, \mathfrak{m}^{K(n)})$, i.e. the following diagram commutes

$$(13.19) \quad \begin{array}{ccc} R_\infty & \xrightarrow{\quad} & \bar{R}_n \\ \downarrow = & & \downarrow \pi_n \\ R_\infty & \xrightarrow{\quad} & \pi_0 \mathcal{R}_S/(p^n, \mathfrak{m}^{K(n)}). \end{array}$$

– this can be done since the right vertical map π_n from (13.18) is surjective. In fact, $R_\infty \rightarrow \bar{R}_n$ can and will be chosen to be *surjective*.¹³

Lemma 13.4. *Suppose $A \xrightarrow{e} B \rightarrow k$ are two Artin local rings over k , both abstractly quotients of $W[[x_1, \dots, x_k]]$. Then any surjection $\varphi : W[[x_1, \dots, x_k]] \rightarrow B$ can be lifted to a surjection $W[[x_1, \dots, x_k]] \rightarrow A$. (All the maps here are assumed to commute with the augmentations to k).*

Proof. First note that a corresponding assertion for vector spaces is easy: if $V \twoheadrightarrow W$ is a surjection of vector spaces, both of dimension $\leq k$, then any spanning set of W of size k can be lifted to a spanning set for V . In terms of matrices, this says that a $(\dim W) \times k$ matrix of maximal rank can be extended to a $(\dim V) \times k$ matrix of maximal rank, which is clear.

We can replace A, B by $A/p, B/p$: if we can lift from B/p to A/p , we get a homomorphism $W[[x_1, \dots, x_k]] \rightarrow A' = A/p \times_{B/p} B$. The map $A \rightarrow A'$ induces an isomorphism $A/p \simeq A'/p$, and is therefore surjective; so any surjection $W[[x_1, \dots, x_k]] \rightarrow A'$ can be lifted to A , and the resulting homomorphism is still surjective.

Let $\mathfrak{m}_A, \mathfrak{m}_B$ be the respective maximal ideals and let I be the kernel of $A \rightarrow B$. A homomorphism $W[[x_1, \dots, x_k]] \rightarrow A$ is now surjective if and only if the images of x_i generate $\mathfrak{m}_A/\mathfrak{m}_A^2$ as a k -vector space, and the same is true for B . We have a short exact sequence $I \hookrightarrow \mathfrak{m}_A \twoheadrightarrow \mathfrak{m}_B$ and so, applying $\otimes_A k$, we get

$$I \otimes_A k \rightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2,$$

exact at right and middle; thus any element in the kernel of $\mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ can be lifted to an element of I .

Now let $y_1, \dots, y_k \in \mathfrak{m}_B$ be the images of x_i in B . Let s_1, \dots, s_k be a generating set for $\mathfrak{m}_A/\mathfrak{m}_A^2$ so that each s_i maps to the class $y_i + \mathfrak{m}_B^2$. Lift y_i arbitrarily to some $y_i^* \in \mathfrak{m}_A$; since the class of $y_i^* - s_i$ dies in $\mathfrak{m}_B/\mathfrak{m}_B^2$, we can, by the above, modify y_i^* to another lift $y_i' \in \mathfrak{m}_A$ of y_i , but such that $y_i' + \mathfrak{m}_A^2 = s_i$. Sending x_i to y_i' gives the desired lift $W[[x_1, \dots, x_k]] \twoheadrightarrow A$. \square

¹³We can also avoid this point, by not choosing $R_\infty \rightarrow \pi_0 \mathcal{R}_S$ at the start, instead just choosing the map $R_\infty \rightarrow \bar{R}_n$ and then using a compactness argument to pass to a subsequence where the resulting maps $R_\infty \rightarrow \pi_0 \mathcal{R}_S/(p^n, \mathfrak{m}^{K(n)})$ converge.

Recall that $S_\infty^\circ = W(k)[[x_1, \dots, x_s]]$. Also fix isomorphisms as in (13.11) for each n , i.e. we fix a \mathbb{Z}/p^n basis for Δ_n . Then there is a map

$$S_\infty^\circ \twoheadrightarrow S_n^\circ,$$

sending x_i to $[\sigma_i] - [e]$, where σ_i is a generator for the i th factor of (\mathbb{Z}/p^n) and e is the identity element. This induces an isomorphism

$$S_\infty^\circ/\mathfrak{a}_n \xrightarrow{\sim} \overline{S}_n^\circ,$$

with \mathfrak{a}_n as in (13.3). Having fixed this, there are also unique maps $S_n^\circ \rightarrow S_m^\circ$ and $\overline{S}_n^\circ \rightarrow \overline{S}_m^\circ$ for $n > m$, compatible with the maps from S_∞° , and the natural map

$$S_\infty^\circ \xrightarrow{\sim} \varprojlim \overline{S}_n^\circ$$

is an isomorphism. Notice that the maps of this paragraph are “meaningless,” i.e. do not correspond to any “map of natural origin,” as they depended on fixing a choice of isomorphisms in (13.11).

Next, for each n we can choose a lift ι_n in the following diagram:

$$(13.20) \quad \begin{array}{ccc} S_\infty^\circ & \xrightarrow{\iota_n} & R_\infty/(p^n, \mathfrak{m}^{K(n)}) \\ \downarrow & & \downarrow \\ \overline{S}_n^\circ & \longrightarrow & \overline{R}_n \end{array}$$

again, this is possible because R_∞ surjects to \overline{R}_n . (Here, \mathfrak{m} is the maximal ideal of R_∞ ; we will allow ourselves to use \mathfrak{m} for the maximal ideals of different local rings when the correct ring is clear by context.)

13.9. Passage to the limit. We now can “pass to the limit” by a compactness argument. Indeed, our discussion to date shows that we get, for each n , the following type of “level n datum”:

- i. A complex C_n of free $S_\infty^\circ/\mathfrak{a}_n$ -modules of bounded rank and bounded support (both bounds being independent of n).
- ii. An action of R_∞ by endomorphisms of C_n , considered as an object in the derived category of $S_\infty^\circ/\mathfrak{a}_n$ -modules. By Lemma 13.3, this action is automatically trivial on $(p^n, \mathfrak{m}^{K(n)})$, where $K(n)$ is as in the quoted Lemma.
- iii. A homomorphism $\iota_n : S_\infty^\circ \rightarrow R_\infty/(p^n, \mathfrak{m}^{K(n)})$ with the property that the two actions (“actions” in the same sense as (ii)) of S_∞° on C_n coincide:
 - via $S_\infty^\circ \rightarrow R_\infty/(p^n, \mathfrak{m}^{K(n)})$ and the action specified in (ii), which automatically factors through $R_\infty/(p^n, \mathfrak{m}^{K(n)})$;
 - via $S_\infty^\circ \rightarrow S_\infty^\circ/\mathfrak{a}_n$ and the fact that C_n is a complex of $S_\infty^\circ/\mathfrak{a}_n$ -modules
- iv. (Recovering the homology of the base manifold): An identification of

$$(13.21) \quad C_n \otimes_{S_\infty^\circ/\mathfrak{a}_n} W_n \simeq C(Y_0, W_n)_\mathfrak{m},$$

in the derived category of W_n -modules, compatibly for the actions of R_∞ . Here $C(Y_0, W_n)_m$ is the complex constructed in §13.3 which computes the homology $H_*(Y_0, W_n)_m$, and the action on the right being via $R_\infty \twoheadrightarrow \pi_0 \mathcal{R}_S$, the surjection chosen at the very beginning of the argument).

Each such datum of level n induces a datum of level n' for each $n' \leq n$: replace C_n by $E_{n'} := C_n \otimes_{S_\infty^\circ/\mathfrak{a}_n} S_\infty^\circ/\mathfrak{a}_{n'}$. The R_∞ -module structure is induced by functoriality, regarding $\otimes_{S_\infty^\circ/\mathfrak{a}_n} S_\infty^\circ/\mathfrak{a}_{n'}$ as a functor between the derived categories, i.e. $r \in R_\infty$ acts on $E_{n'}$ as “ $r \otimes 1$.” $\iota_{n'}$ is obtained by projecting ι_n .

It is easy to see that there are only finitely many possible data of level n up to isomorphism. Therefore, by a compactness argument, there is a subsequence j_n of integers so that the data associated to (Q_{j_n}, n) are all compatible – i.e.:

- (Q_{j_n}, n) is the datum of level n obtained from the level j_n datum Q_{j_n} , in the fashion just described;
- The datum of level $n - 1$ induced by (Q_{j_n}, n) is actually isomorphic to the datum associated to $(Q_{j_{n-1}}, n - 1)$.

13.10. Reindexing. As we just saw, by passing to the subsequence j_n we can achieve a compatible system of data, as above. We will now change notation somewhat so we don’t have to keep writing j_n s.

Let us talk this through carefully now and then we will simplify the notation; when, later on, we need to consider more carefully the internals of the limit process, we will refer back to this section. (The reader might want to skip this somewhat tedious discussion, and just pretend that $j_n = n$).

For any $n \leq m$, we can regard \mathfrak{a}_n as an ideal of $\overline{S_m^\circ}$ by means of the map $S_\infty^\circ \rightarrow \overline{S_m^\circ}$. The resulting ideal doesn’t depend on the choice of this map: it is the ideal of $\overline{S_m^\circ} = W_m[\Delta_m]$ given by the kernel of the map $W_m[\Delta_m] \rightarrow W_n[\Delta_m/p^n]$.

Set

$$D_n := C_{j_n} \otimes_{S_\infty^\circ/\mathfrak{a}_{j_n}} S_\infty^\circ/\mathfrak{a}_n,$$

a complex of free $S_\infty^\circ/\mathfrak{a}_n$ modules – this is part of the level n datum obtained from Q_{j_n} .

It comes with compatible actions of R_∞ on D_n , as an object in the derived category of $S_\infty^\circ/\mathfrak{a}_n$; also the homomorphisms obtained by composing

$$S_\infty^\circ \rightarrow R_\infty/(p^{j_n}, \mathfrak{m}^{K(j_n)}) \rightarrow R_\infty/(p^n, \mathfrak{m}^{K(n)})$$

are compatible with one another as n increases. (The first map was the one chosen at step j_n of the limit process, the second map is the obvious one). Therefore, passing to the $n \rightarrow \infty$ limit, we get

$$(13.22) \quad \iota : S_\infty^\circ \rightarrow R_\infty$$

whose reduction modulo $(p^n, \mathfrak{m}^{K(n)})$ recovers the maps above.

Next, set

$$(13.23) \quad \Delta_n^{\text{reindexed}} = \Delta_{j_n}/p^n$$

$$(13.24) \quad \overline{S}_n^{\text{reindexed}} = \underbrace{S_{j_n}^\circ / \mathfrak{a}_n}_{W_{j_n}[\Delta_{j_n}]/\mathfrak{a}_n} = W_n[\Delta_n^{\text{reindexed}}]$$

$$(13.25) \quad \overline{R}_n^{\text{reindexed}} = \left(\overline{R}_{j_n} \otimes_{\overline{S}_{j_n}^\circ} \overline{S}_{j_n}^\circ / \mathfrak{a}_n \right) / \mathfrak{m}^{K(n)}$$

Here, we regard \mathfrak{a}_n as an ideal of $\overline{S}_{j_n}^\circ$ as noted above. We introduce the superscript “reindexed” to recall that this has been reindexed after the limit process. Just as before, the choice of isomorphisms in (13.11) give an identification of $\overline{S}_n^{\text{reindexed}}$ with $S_\infty^\circ / \mathfrak{a}_n$, and thus make $\overline{S}_n^{\text{reindexed}}$ into an inverse system with limit S_∞° .

Note that deformations classified by $\overline{R}_n^{\text{reindexed}}$ are deformations of level $S \coprod Q_{j_n}$, but forcing the inertial level ((8.10)) to be $\leq n$ at primes in Q_{j_n} . The morphism $R_\infty \rightarrow \overline{R}_{j_n}$ chosen in the limit process induces $R_\infty \rightarrow \overline{R}_n^{\text{reindexed}}$ and the action of R_∞ on D_n factors through $\overline{R}_n^{\text{reindexed}}$ by the definitions.

Remark. Although we don’t use it in this paper, we observe that D_n has a “physical” interpretation: it is naturally identified with a complex that computes the homology of $Y(Q_{j_n}, n)$. More precisely, D_n is naturally identified with $C_{j_n} \otimes_{W_{j_n}[\Delta_{j_n}]} W_n[\Delta_n^{\text{reindexed}}]$. The natural projection $Y(Q_{j_n}, j_n) \rightarrow Y(Q_{j_n}, n)$ induces a quasi-isomorphism of the latter with the complex $C_*^{\Delta_n^{\text{reindexed}}}(Y(Q_{j_n}, n), W_n)_{\mathfrak{m}'}$, where \mathfrak{m}' is defined similarly to (13.12). Similarly, the identification $C_{j_n} \otimes_{\overline{S}_{j_n}^\circ} W_{j_n} \simeq C(Y_0, W_{j_n})_{\mathfrak{m}}$ from (13.13) induces a similar identification $D_n \otimes_{\overline{S}_n^{\text{reindexed}}} W_n \simeq C(Y_0, W_n)_{\mathfrak{m}}$.

We finally observe that the map $R_\infty \rightarrow \overline{R}_n^{\text{reindexed}}$ can be extended to a map

$$(13.26) \quad R_\infty \otimes_{S_\infty^\circ} \overline{S}_n^{\text{reindexed}} \rightarrow \overline{R}_n^{\text{reindexed}}.$$

(Later we will see this is actually an isomorphism, under our assumptions.) In fact, we have a commutative diagram

$$(13.27) \quad \begin{array}{ccccc} S_\infty^\circ & \xrightarrow{\tau} & R_\infty / (p^n, \mathfrak{m}^{K(n)}) & & \\ \downarrow & & \downarrow & \searrow & \\ \overline{S}_n^{\text{reindexed}} & \longrightarrow & \overline{R}_n^{\text{reindexed}} & \longrightarrow & \pi_0 \mathcal{R}_S / (p^n, \mathfrak{m}^{K(n)}) \end{array}$$

where the composite $\overline{S}_n^{\text{reindexed}} \rightarrow \pi_0 \mathcal{R}_S / (p^n, \mathfrak{m}^{K(n)})$ at the bottom factors through the natural augmentation $\overline{S}_n^{\text{reindexed}} \rightarrow W_n$. The commutativity of the left-hand square follows from (13.20) and the commutativity of the right hand triangle follows similarly from (13.19).

Henceforth we drop the superscript “reindexed” from the notation; when we need to be explicit, we refer back to this section.

13.11. Analysis of the limit. After reindexing, as just described, we have a sequence of complexes D_n of free \overline{S}_n° -modules, together with isomorphisms

$$(13.28) \quad D_n \otimes_{\overline{S}_n^\circ} \overline{S}_m^\circ \xrightarrow{\sim} D_m$$

Set $D_\infty = \varprojlim D_n$ (where the transition maps are those from (13.28)). It is now a complex of free modules under $\varprojlim \overline{S}_n^\circ = S_\infty^\circ$. Because all the groups involved are finite, we have

$$H_* D_\infty \cong \varprojlim H_* D_n.$$

Now R_∞ acts on each $H_*(D_n)$ and these actions are compatible with the maps $H_*(D_n) \rightarrow H_*(D_m)$ for $n > m$; therefore, we obtain a R_∞ action on $H_* D_\infty$ too. In fact, we can even get a R_∞ -action on D_∞ in the derived category of S_∞° -modules using [20, Lemma 2.13]: The generators $x_1, \dots, x_{s-\delta}$ of R_∞ are only homotopy compatible at each level, but this is nonetheless just enough to lift them to D_∞ in a fashion that is unique up to homotopy; the uniqueness means that the resulting lifts also commute up to homotopy, thus giving an action of R_∞ . Thus we have a quasi-isomorphism $D_\infty \otimes_{S_\infty^\circ} \overline{S}_n^\circ \xrightarrow{\sim} D_n$, compatible with R_∞ -actions.

Finally, we have as in (13.22) a limit map $\iota = \varprojlim \iota_n : S_\infty^\circ \rightarrow R_\infty$. By assumption (iii) of §13.9 the two actions of S_∞° on $H_* D_n$ – one via ι_n and one via $S_\infty^\circ \rightarrow \overline{S}_n^\circ$ – coincide, and by passage to the limit, the two actions of S_∞° on $H_* D_\infty$ – one via ι and the other the natural one – coincide; in fact the two actions on D_∞ coincide in the derived category, by the uniqueness in [20, Lemma 2.13].

The key lemma in commutative algebra that follows shows that D_∞ is a resolution of the edge homology, i.e. $H_i D_\infty = 0$ for $i > q$, and so D_∞ is quasi-isomorphic to $M := H_q D_\infty$, shifted in degree q :

Lemma 13.5. (*Calegari–Geraghty, Hansen*) *Suppose given a complex D_∞ of finite free S_∞° modules, supported in degrees $[q, q + \delta]$ over $S_\infty^\circ \cong W[[x_1, \dots, x_s]]$, with degree-decreasing differential. Suppose also we are given a homomorphism $S_\infty^\circ \rightarrow R_\infty \cong W(k)[[x_1, \dots, x_{s-\delta}]]$ and the action of S_∞° on $H_* D_\infty$ factors through this homomorphism. Then $H_i D_\infty$ is nonzero only for $i = q$, and $H_q D_\infty$ is free over R_∞ .*

Proof. ([6, Lemma 3.2], [20, Lemma 2.9]): Let j be the largest integer for which $H_{q+j} D_\infty \neq 0$. Say $j = \delta - \delta'$, so that

$$D_{\infty, q+\delta-\delta'} \leftarrow \dots \leftarrow D_{\infty, q+\delta}$$

is a S_∞° -projective resolution of certain overmodule $\tilde{M} \supset H_{q+j} D_\infty$. By the Auslander-Buchsbaum formula, $\text{depth}_{S_\infty^\circ}(\tilde{M}) + \text{proj.dim}_{S_\infty^\circ}(\tilde{M}) = \dim(S_\infty^\circ)$, and so the S_∞° -depth of \tilde{M} is at least $\dim S_\infty^\circ - \delta + \delta'$.

Therefore, the dimension of $H_{q+j} D_\infty$ as an S_∞° -module is at least $\dim S_\infty^\circ - \delta + \delta'$ (a lemma from commutative algebra: depth bounds from below the dimension of any submodule).

But if N is a module over R_∞ which is finitely generated when considered as an S_∞° -module, then $\dim_{S_\infty^\circ} N \leq \dim R_\infty$. This is Lemma 2.8 (3) of Khare-Thorne [20]; note that our usage of S and R is switched from theirs.

So we have a contradiction unless $\delta' = 0$, i.e. D_∞ is a projective resolution of $H_q D_\infty$; also the dimension and depth of $H_q D_\infty$ as an S_∞° -module was $\dim(R_\infty)$.

In particular, the R_∞ -depth of $H_q D_\infty$ is $\dim(R_\infty)$ (definition via regular sequences); Auslander–Buchsbaum now forces $H_q D_\infty$ to be free over R_∞ . \square

Thus, if M is the unique nonvanishing homology group of D_∞ , placed in degree q , there is a natural quasi-isomorphism $D_\infty \xrightarrow{\sim} M$, compatible (by definition) with the R_∞ -actions and we have quasi-isomorphisms in the derived category of \overline{S}_n° -modules:

$$(13.29) \quad \underbrace{M \otimes_{S_\infty^\circ} \overline{S}_n^\circ}_{\text{derived, i.e. Tor}} \xleftarrow{\sim} D_\infty \otimes_{S_\infty^\circ} \overline{S}_n^\circ \xrightarrow{\sim} D_n.$$

These are compatible with the action of R_∞ , where $r \in R_\infty$ is acting as $r \otimes 1$ on the first and second terms in the sequence. (This follows from the definition of “compatible data.”) In particular, we get

$$(13.30) \quad \underbrace{M \otimes_{S_\infty^\circ} \overline{S}_n^\circ}_{\text{usual tensor product (not Tor)}} \cong H_q(D_n),$$

compatibly with the action of R_∞ .

Finally return to (13.26):

$$R_\infty \otimes_{S_\infty^\circ} \overline{S}_n^\circ \twoheadrightarrow \overline{R}_n.$$

We claim it is an *isomorphism*. Indeed, writing as usual \mathfrak{a}_n for the kernel of $S_\infty \rightarrow \overline{S}_n^\circ$, we must prove that

$$(13.31) \quad R_\infty / \iota(\mathfrak{a}_n) R_\infty \xrightarrow{\sim} \overline{R}_n.$$

Clearly that map is surjective, so we just need to prove injectivity. Suppose $x \in R_\infty$ maps to zero in \overline{R}_n . Then x acts trivially on $H_0 D_n$ (discussion after (13.25)). But in (13.30) we see that $H_0 D_n$ is identified, as a module for R_∞ , with the quotient $M/\mathfrak{a}_n M$, and M is free over R_∞ ; thus $x \in \iota(\mathfrak{a}_n) R_\infty$ as desired.

Finally, note that we also can draw a conclusion about the homology of Y_0 itself. We have an identification:

$$\begin{aligned} H_*(Y_0, W)_m &= \varprojlim_n H_*(Y_0, W_n)_m \stackrel{(13.21)}{\simeq} \varprojlim_n H_* \left(D_n \otimes_{\overline{S}_n^\circ} W_n \right) \\ &= H_* \left(\varprojlim_n D_n \otimes_{\overline{S}_n^\circ} W_n \right) = H_*(D_\infty \otimes_{S_\infty^\circ} W) \xrightarrow{\sim} \mathrm{Tor}_*^{S_\infty^\circ}(M, W), \end{aligned}$$

which carries a free action of $\mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W)$, as desired.

In particular, in minimal nonvanishing degree, we obtain an identification

$$(13.32) \quad H_{\min}(Y_0, W)_m \cong M \otimes_{S_\infty^\circ} W$$

compatible with the action of R_∞ (acting on the right as $r \otimes 1$).

The surjection $R_\infty \rightarrow \pi_0 \mathcal{R}_S$ fixed at the start of the argument factors through $R_\infty \otimes_{S_\infty^\circ} W$: if $a \in S_\infty^\circ$ lies in the kernel of the augmentation $S_\infty^\circ \rightarrow W$, then $\iota(a)$ dies in $\pi_0 \mathcal{R}_S / (p^n, \mathfrak{m}^{K(n)})$ for all n , by diagram (13.27). Comparing this with (13.32) we see that

$$R_\infty \otimes_{S_\infty^\circ} W \twoheadrightarrow \pi_0 \mathcal{R}_S \twoheadrightarrow \text{image of Hecke algebra on } H_{\min}(Y_0, W)_{\mathfrak{m}}$$

are both isomorphisms.

This concludes the proof of Theorem 13.1 – taking for the Q s the sets Q_{j_n} , considered as allowable Taylor–Wiles data of level n , for D_∞ we take D_∞ . For the \bar{R}_n s (as in the theorem statement) we take \bar{R}_n as above, which were explicitly given during the re-indexing process of §13.10.

14. IDENTIFICATION OF $\pi_* \mathcal{R}_S$

Continuing in the situation of §13.1 – in particular, the Galois representation associated to a cohomology class on \mathbf{G} – we can identify $\pi_* \mathcal{R}_S$ as an abstract graded ring. Again we recall our standing assumption that when we speak of global Galois deformation rings, we are always imposing the crystalline condition.

Theorem 14.1. *Notations and assumptions as in §13.1, and assume Conjecture 6.1. Also use the notation of Theorem 13.1. Then there is an isomorphism of graded rings*

$$(14.1) \quad \pi_* \mathcal{R}_S \cong \mathrm{Tor}_{S_\infty^\circ}^*(R_\infty, W),$$

between the associated graded ring $\pi_ \mathcal{R}_S$ to the derived deformation ring \mathcal{R}_S at base level, and the Tor-algebra on the right.*

In particular, using Theorem 13.1 (e), the homology $H_(Y_0, W)_{\mathfrak{m}}$ carries the structure of a free graded module over the graded ring $\pi_* \mathcal{R}_S$.*

This result seems rather unsatisfying at first, since it does not pin down any characterizing property of the isomorphism or the action. It must be so, in a sense, because everything depends on the limit process in the Taylor–Wiles method.

However, even at this abstract level, the following consequence of the final sentence is interesting:

The homotopy groups $\pi_j \mathcal{R}_S$ are vanishing unless $j \in [0, \delta]$.

In the final section of this paper we will show that the action of $\pi_* \mathcal{R}_S$ on $H_*(Y_0, W)_{\mathfrak{m}}$ from the statement – more precisely, the explicit action that is constructed in the proof of the statement – is closely related to the action of a derived Hecke algebra, and is in particular independent of all choices made during the proof.

Also, note that the Theorem *does not use* the full strength of the assumptions from §13.1; in particular, it doesn't use assumption 7(c) that excludes congruences with other forms.

Proof. (of the Theorem)

Apply Theorem 13.1; it gives Taylor–Wiles sets Q_n , limit rings R_∞, S_∞ , and isomorphisms from R_∞/\mathfrak{a}_n to a quotient \overline{R}_n of the usual deformation ring at level $S \coprod Q_n$ (plus more!)

We next proceed as in §11.3, and use notation as there: \mathcal{R}_n is the derived deformation ring at level $S \coprod Q_n$, \mathcal{S}_n is the framed derived deformation ring for places in Q_n , and $\mathcal{S}_n^{\text{ur}}$ is the unramified version of \mathcal{S}_n .

The diagram $\mathcal{R}_n \leftarrow \mathcal{S}_n \rightarrow \mathcal{S}_n^{\text{ur}}$ maps to the diagram of discrete rings

$$(14.2) \quad \overline{R}_n \leftarrow \underbrace{\pi_0 \mathcal{S}_n \otimes_{W[\tilde{\Delta}_n]} W_n[\Delta_n]}_{:= \overline{S}_n} \rightarrow \underbrace{\pi_0 \mathcal{S}_n^{\text{ur}} / p^n}_{:= \overline{S}_n^{\text{ur}}}$$

Here, we recall from §13.6 that $\tilde{\Delta}_n = \prod_{q \in Q_n} \mathbf{T}(\mathbb{F}_q)_p$ is the Galois group of the covering $Y_1(Q_n) \rightarrow Y_0(Q_n)$, whereas $\Delta_n = \tilde{\Delta}_n / p^n$ is just the piece of it used in the limit process; the map from $W[\tilde{\Delta}_n]$ to $\pi_0 \mathcal{S}_n$ is that arising from the description of the latter as a group algebra (Remark 8.7).

Note that \overline{S}_n is larger than the ring \overline{S}_n° – roughly the former measures all local deformations at Q_n whereas the latter just measures deformations on inertia. For example, if Q_n consisted only of the single prime q , refer to (8.14) to see the difference (the bars over \overline{S}_n and \overline{S}_n° are because they are reduced modulo p^n). However, the diagram (14.2) still admits a map from

$$(14.3) \quad \overline{R}_n \leftarrow \overline{S}_n^\circ \rightarrow W_n.$$

This map of diagrams, from (14.3) to (14.2), induces a weak equivalence on derived tensor product (by this, we mean as usual a weak equivalence of represented functors) by (8.17), or its obvious analogue with more than one prime.

Thus, we get

$$(14.4) \quad \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \xrightarrow{(11.4)} \mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\text{ur}} \rightarrow \overline{R}_n \otimes_{\overline{S}_n} \overline{S}_n^{\text{ur}} \xleftarrow{\sim} \overline{R}_n \otimes_{\overline{S}_n^\circ} W_n \xrightarrow{(13.5)} R_\infty / \mathfrak{a}_n \otimes_{S_\infty^\circ / \mathfrak{a}_n} W_n.$$

By the same type of discussion as (7.7), we can invert the weak equivalence above in a homotopical sense, obtaining a map

$$\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \longrightarrow R_\infty / \mathfrak{a}_n \otimes_{S_\infty^\circ / \mathfrak{a}_n} W_n.$$

in pro-Art_k which behaves in the expected way on tangent complexes. We will apply Theorem 12.1 to these maps. Once we check the conditions of this Theorem, it proves Theorem 14.1.

To apply Theorem 12.1, we must check two conditions: a numerical condition (12.1) and a tangent space condition (12.3). (12.1) follows from Tate’s Euler characteristic formula (this is quite routine – for a closely related computation, see §13.7). So let us examine (12.3); it will basically follow from Theorem 11.1, but let us write out the details.

We must study the composite

$$(14.5) \quad \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \rightarrow R_{\infty}/\mathfrak{a}_n \otimes_{S_{\infty}^{\circ}/\mathfrak{a}_n} W_n \rightarrow R_{\infty}/\mathfrak{a}_m \otimes_{S_{\infty}^{\circ}/\mathfrak{a}_m} W_m$$

for $n \geq m$: we must show it is an isomorphism on \mathfrak{t}^0 and a surjection on \mathfrak{t}^1 .

To study (14.5) consider, for $n \geq m$, the quotients

$$(14.6) \quad \pi_0 \mathcal{R}_n \twoheadrightarrow \overline{R}_{n,m}, \pi_0 \mathcal{S}_n \twoheadrightarrow \overline{S}_{n,m}, \pi_0 \mathcal{S}_n^{\text{ur}} \twoheadrightarrow \overline{S}_{n,m}^{\text{ur}}$$

defined thus: take $\overline{R}_{n,m} = \overline{R}_n/\mathfrak{a}_m$, $\overline{S}_{n,m} = \overline{S}_n/\mathfrak{a}_m$ and $\overline{S}_{n,m}^{\text{ur}}$ to be the quotient $\overline{S}_n^{\text{ur}}/p^m$. Here \mathfrak{a}_m in each case means the image of the ideal $\mathfrak{a}_m \subset S_{\infty}^{\circ}$ (see (13.3)); note there is a map $S_{\infty}^{\circ} \xrightarrow{(13.4)} \overline{S}_n^{\circ} = W_n[\Delta_n] \xrightarrow{(8.14)} \overline{S}_n$.

All these maps are isomorphisms on \mathfrak{t}^0 , at least if $m \geq 2$ (they are automatically injective and then one can count dimensions; and for $m \geq 2$ the ideals \mathfrak{a}_m and p^m are contained in the square of the maximal ideal of S_{∞}° and thus do not affect tangent spaces).

Theorem 11.1 shows that the resulting map

$$\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \rightarrow \overline{R}_{n,m} \otimes_{\overline{S}_{n,m}} \overline{S}_{n,m}^{\text{ur}}$$

is an isomorphism on \mathfrak{t}^0 and a surjection on \mathfrak{t}^1 . But then the same assertion holds for (14.5), as follows from the diagram

$$(14.7) \quad \begin{array}{ccccc} \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & R_{\infty}/\mathfrak{a}_n \otimes_{S_{\infty}^{\circ}/\mathfrak{a}_n} W_n & \longrightarrow & R_{\infty}/\mathfrak{a}_m \otimes_{S_{\infty}^{\circ}/\mathfrak{a}_m} W_m \\ \downarrow = & & \downarrow (13.5) & & \downarrow t \\ \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & \overline{R}_n \otimes_{\overline{S}_n} \overline{S}_n^{\text{ur}} & \longrightarrow & \overline{R}_{n,m} \otimes_{\overline{S}_{n,m}} \overline{S}_{n,m}^{\text{ur}} \end{array}$$

Here, the right hand vertical arrow t induces an isomorphism on tangent complexes, for reasons similar to the validity of (8.17). \square

14.1. During the proof of this Theorem, we carried out two limit processes: first the limit process of Theorem 13.1 and then again the limit process of Theorem 12.1. By a slight reindexing, very similar to that already done in §13.10, it is possible to set things up so these limit processes apply simultaneously - this is a minor observation that will be helpful in the next section.

Explicitly, it is possible to choose the sets Q_n in such a way that:

- the conclusion of Theorem 13.1 remains true, *and*
- the composite maps $f_n : \mathcal{R}_S \rightarrow R_{\infty}/\mathfrak{a}_n \otimes_{S_{\infty}^{\circ}/\mathfrak{a}_n} W_n$ from (14.4) all satisfy the conditions of Theorem 12.1, *and*
- These composite maps are all homotopy compatible, in the sense of the proof of Theorem 12.1 – in particular, they fit together to give an isomorphism

$$(14.8) \quad \pi_* \mathcal{R}_S \cong \text{Tor}_{S_{\infty}^{\circ}}(R_{\infty}, W).$$

This is just like §13.10: first choose sets Q_n as in Theorem 13.1, then apply Theorem 12.1. What the proof of Theorem 12.1 actually outputs is a subsequence j_n so that the composite of f_{j_n} with the natural map $R_\infty/\mathfrak{a}_{j_n} \otimes_{S_\infty/\mathfrak{a}_{j_n}} W_{j_n} \rightarrow R_\infty/\mathfrak{a}_n \otimes_{S_\infty/\mathfrak{a}_n} W_n$. Now reindex just like §13.10: set $Q_n^{\text{reindexed}} = Q_{j_n}$, and let $\overline{R}_n^{\text{reindexed}}$ be the quotient $R_\infty/(\mathfrak{a}_n, \mathfrak{m}^{K(n)})$ of $\overline{R}_{j_n} \cong R_\infty/\mathfrak{a}_{j_n}$.

15. COMPARISON WITH THE DERIVED HECKE ALGEBRA. CONCLUSIONS

The goal of this section is to compare the action of $\pi_* \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ that we have defined, with the action of the derived Hecke algebra from [37]. This comparison (Theorem 15.2) will show, in particular, that the action of $\pi_* \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ constructed in Theorem 14.1 do not depend on the choices involved in that construction. Unfortunately, the constructions from [37] are a little long to describe here. Our current discussion is therefore not at all self-contained: we will freely quote what we need from [37].

Let us note (as mentioned in Remark 1.1) that our fairly strong local assumptions on ρ force $\pi_* \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ to be an *integral* exterior algebra, and the analysis of this section will really use this structure. However, we expect that the analogues of the results of this section will remain valid without these local assumptions, so long as one tensors with \mathbb{Q} .

We continue with the notations that have been set up in prior sections, in particular in §13.1.

15.1. The derived Hecke algebra. In the paper [37] a derived version of the Hecke algebra is introduced; it acts on $H^*(Y_0, W)$ by degree-increasing endomorphisms or (by a slight variant, replacing the role of cup product with cap products) on $H_*(Y_0, W)$ by degree-decreasing endomorphisms. Thus it cannot be “the same as” the action of $\pi_* \mathcal{R}$ because the operations move degrees in different directions.

Our statement is rather that the two actions are “compatible,” in a sense to be explained – if V is a vector space, there is a natural action of $\wedge^* V$ on $\wedge^* V$ by multiplication, but there is *also* an action of $\wedge^* V^*$ by contractions. The relationship between the two actions is a model for the relationship between the derived Hecke algebra and $\pi_* \mathcal{R}$.

15.2. Some linear algebra. Let V be a finite-dimensional vector space, with dual V^* , and let M be a finite-dimensional graded vector space (over the same field) that carries a graded action of $\wedge^* V$ and $\wedge^* V^*$; here V increases degree by 1 and V^* decreases degree by 1.

We say these actions are *compatible* if for $v^* \in V^*$ and $v \in V$ we have

$$(15.1) \quad v^* \cdot v \cdot m + v \cdot v^* \cdot m = \langle v, v^* \rangle \cdot m$$

Thus, the standard actions of $\wedge^* V, \wedge^* V^*$ on $\wedge^* V$ (the first by multiplication, the second by contractions) are compatible.

Suppose now that M is generated as a $\wedge^* V$ -module by elements of minimal degree. In that case, (15.1) uniquely specifies the $\wedge^* V^*$ action. Similarly the other way around.

Now take

$$V = H_f^1(\mathrm{Ad}^* \rho_{\mathcal{O}}(1))^\vee,$$

where the \vee means to take W -linear homomorphisms to $W(k)$, $\rho_{\mathcal{O}}$ is as in (13.2), and Ad^* is the dual of the adjoint representation. Under our assumptions, it is [37, §8.8] a finite free $W(k)$ -module of rank δ ; in [37, Theorem 8.5] we construct, by means of the derived Hecke algebra and under the same conjecture and same local hypotheses on ρ , an action:

$$(15.2) \quad \wedge^* V \hookrightarrow H^*(Y_0, W)_{\mathfrak{m}}$$

By adjointness we obtain a homological version:

$$(15.3) \quad \wedge^* V \hookrightarrow H_*(Y_0, W)_{\mathfrak{m}}$$

which is adjoint to (15.2) (in the case at hand, the natural pairing of homology and cohomology is a perfect pairing with W -coefficients). This can also be constructed directly from an action of the derived Hecke algebra: in [37] the derived Hecke algebra acts on cohomology; but replacing the role of cup products by cap products it also acts on homology.

We will exhibit below (see (15.7)) an isomorphism

$$(15.4) \quad \pi_* \mathcal{R}_S \cong \wedge^* V^*$$

and show that the following actions are compatible, in the sense of (15.1):

- the action of $\wedge^* V$ via (15.3)
- the action of $\wedge^* V^*$ via (15.4) and the action constructed in the course of proving Theorem 14.1

In particular this means that the action of $\pi_* \mathcal{R}_S$ on $H_*(Y_0, W)_{\mathfrak{m}}$ is independent of the choice of Taylor-Wiles sets.

We begin with:

Lemma 15.1. *As usual, let \mathcal{R}_S be the crystalline deformation ring of ρ ; let $A \in \mathrm{Art}_k$ be homotopy discrete, and M a discrete A -module. Fix a lift $\rho_A : \Gamma_S \rightarrow G(A)$, classified by a map $\phi : \pi_0 \mathcal{R}_S \rightarrow A$.*

The set of homotopy classes of maps $\mathcal{R}_S \rightarrow A \oplus M[1]$ lifting ϕ are in natural bijection with $H_f^2(\mathrm{Ad} \rho_A \otimes M)$.

In this statement, $\mathrm{Ad} \rho_A \otimes M$ refers to $\mathrm{Lie}(G)_{W(k)} \otimes M$, endowed with a Γ_S -module structure by identifying it with the kernel of $G(A \oplus M) \rightarrow G(A)$.

Proof. For $A = k$ and $M = k$ this is the computation of the tangent complex, i.e. Theorem 9.2. The same formalism works replacing k by A ; for example if $\mathcal{F} : \mathrm{Art}_k \rightarrow \mathbf{sSets}$ is a formally cohesive functor, and we fix a vertex x_0 of set

$\mathcal{F}(A)$ (where A is homotopy discrete) then we may form an Ω -spectrum whose n th space is

$$X_n = \text{homotopy fiber of } \mathcal{F}(A \oplus M[n]) \rightarrow \mathcal{F}(A) \text{ above } x_0$$

and it has similar formal properties to the tangent complex; the explicit computation in the case at hand proceeds just as in (the various inputs to) Theorem 9.2. \square

Let us note for future reference that we can also describe the π_0 described in the Lemma as the André–Quillen cohomology group

$$(15.5) \quad D_{\mathbb{Z}}^1(\mathcal{R}_S, M),$$

which we understand, for \mathcal{R}_α a pro-system, to be the direct limit $\varinjlim D_{\mathbb{Z}}^1(\mathcal{R}_\alpha, M)$; this description is valid with \mathcal{R}_S replaced by any pro-simplicial ring.

Now, any map $\tilde{\phi} : \mathcal{R}_S \rightarrow A \oplus M[1]$ induces a map $\pi_1 \mathcal{R}_S \rightarrow M$. This, and the statement of the Lemma, gives us a pairing

$$(15.6) \quad \pi_1 \mathcal{R}_S \times H_f^2(\text{Ad} \rho_A \otimes M) \rightarrow M.$$

In particular, taking $A = W_n$, $M = (W \otimes \mathbb{Q})/W$, and the representation ρ_A to be the mod ϖ^n reduction of the representation (13.2) defined by Π , we get (after Tate global duality and passage to the limit)

$$(15.7) \quad \pi_1 \mathcal{R}_S \rightarrow H_f^1(\text{Ad}^* \rho_O(1)) = V^*.$$

We will see later that this map is an isomorphism, and its inverse induces an isomorphism $\wedge^* V^* \rightarrow \pi_* \mathcal{R}_S$.

15.3. Background on the action (15.2). Now and in the remainder of this section, we put ourselves in the situation of §14.1. In other words, we are given a sequence of allowable Taylor–Wiles data such that the limit process of Theorem 13.1 and the limit process of Theorem 12.1 can be carried out simultaneously (see §14.1). In particular, as in Theorem 13.1, we have “limit rings,” augmented to $W(k)$:

$$S_\infty^\circ \xrightarrow{\iota} R_\infty \twoheadrightarrow W(k).$$

Let \mathfrak{p}_S be the kernel of the augmentation $S_\infty^\circ \rightarrow W(k)$ (and similar for R_∞); write \mathfrak{t}_S^* and \mathfrak{t}_R^* for the quotient space $\mathfrak{p}_S/\mathfrak{p}_S^2$. Let $\mathfrak{t}_S, \mathfrak{t}_R$ be the W -linear duals. To say differently,

$$\mathfrak{t}_S = D_W^0(S_\infty^\circ, W)$$

is the set of derivations of the W -algebra S_∞° with values in W and similarly for R_∞ .

The map $S_\infty^\circ \twoheadrightarrow R_\infty$ (recall it is surjective, discussion after Theorem 13.1) induces $\mathfrak{t}_R \hookrightarrow \mathfrak{t}_S$, and we write $\mathfrak{t}_S/\mathfrak{t}_R$ be the cokernel. In suitable coordinates we have ([37, §7.3]):

$$S_\infty \cong W[[x_1, \dots, x_s]], R_\infty \cong W[[y_1, \dots, y_{s-\delta}]]$$

and the map between them sends x_i to y_i for $i \leq s - \delta$ and kills x_i for $i > s - \delta$.

There are natural isomorphisms

$$(15.8) \quad \text{Ext}_{S_\infty}^1(W, W) \cong \mathfrak{t}_S$$

$$(15.9) \quad \mathrm{Tor}_1^{\mathrm{S}_\infty}(\mathrm{R}_\infty, W) \cong (\mathfrak{t}_S/\mathfrak{t}_R)^*$$

Indeed the sequence $\mathfrak{p}_S \rightarrow \mathrm{S}_\infty \rightarrow W$ induces $\mathrm{Ext}_{\mathrm{S}_\infty}^1(W, W) \cong \mathrm{Hom}_{\mathrm{S}_\infty\text{-mod}}(\mathfrak{p}_S/\mathfrak{p}_S^2, W)$. Similarly write K for the kernel of $\mathrm{S}_\infty \rightarrow \mathrm{R}_\infty$; the sequence $K \rightarrow \mathrm{S}_\infty \rightarrow \mathrm{R}_\infty$ of S_∞ -modules induces $\mathrm{Tor}_1^{\mathrm{S}_\infty}(\mathrm{R}_\infty, W) = K \otimes_{\mathrm{S}_\infty} W = (K/\mathfrak{p}_S K)$; the inclusion $K \hookrightarrow \mathfrak{p}_S$ maps this isomorphically to the kernel of $\mathfrak{p}_S/\mathfrak{p}_S^2 \rightarrow \mathfrak{p}_R/\mathfrak{p}_R^2$, giving rise to an isomorphism $\mathrm{Tor}_1^{\mathrm{S}_\infty}(\mathrm{R}_\infty, W) \cong (\mathfrak{t}_S/\mathfrak{t}_R)^*$.

The isomorphisms (15.8) and (15.9) induce:

$$(15.10) \quad \mathrm{Ext}_{\mathrm{S}_\infty}^*(W, W) \cong \wedge^* \mathfrak{t}_S$$

$$(15.11) \quad \mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{R}_\infty, W) \cong \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)^*$$

More precisely, in both cases, one computes that the left-hand side with its natural algebra structure is a free exterior algebra on its degree 1 component.

Now Theorem 13.1 part (d) gives an identification

$$(15.12) \quad H_*(Y_0, W)_\mathfrak{m} \cong \mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{D}_\infty, W) \cong \mathrm{Tor}_*^{\mathrm{S}_\infty}(\underbrace{H_q(\mathrm{D}_\infty)[-q]}_M, W)$$

where D_∞ is homologically concentrated in degree q and $M := H_q(\mathrm{D}_\infty)$ is a free module over R_∞ . $\mathrm{Tor}_*^{\mathrm{S}_\infty}(M, W)$ has both a natural structure of a graded $\mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{R}_\infty, W) \cong \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)^*$ -module and also a natural structure of module under $\mathrm{Ext}_{\mathrm{S}_\infty}^*(W, W) \cong \wedge^* \mathfrak{t}_S$. The latter action factors through $\wedge^* \mathfrak{t}_S \rightarrow \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)$. The resulting actions of $\wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)$ and $\wedge^*(\mathfrak{t}_S/\mathfrak{t}_R)^*$ are compatible, in the sense of (15.1). All these assertions are verified by explicit computation.

Now given a “convergent” sequence of Taylor–Wiles data as in the statement of Theorem 14.1, yields an identification [37, §8.25, proof of Theorem 8.5]

$$(15.13) \quad \mathfrak{t}_S/\mathfrak{t}_R \cong \mathbf{V}$$

such that the following diagram commutes:

$$(15.14) \quad \begin{array}{ccc} \mathrm{Ext}_{\mathrm{S}_\infty}(W, W) & \xrightarrow{\quad} & \mathrm{End}(\mathrm{Tor}_*^{\mathrm{S}_\infty}(M, W)) \\ (15.10) \downarrow & & (15.12) \downarrow \sim \\ \wedge^*(\mathfrak{t}_S/\mathfrak{t}_R) & \xrightarrow[(15.13)]{\quad} \wedge^* \mathbf{V} & \xrightarrow{(15.3)} \mathrm{End}(H_*(Y_0, W)_\mathfrak{m}) \end{array}$$

(of course, in this diagram, the identification (15.12) also depended on the choices of Theorem 14.1).

We are now ready for the basic result relating the derived Hecke algebra with the action of the derived deformation ring:

Theorem 15.2. *The actions of $\wedge^* \mathbf{V}$ on $H_*(Y_0, W)_\mathfrak{m}$ via (15.3) and $\wedge^* \mathbf{V}^*$ on $H_*(Y_0, W)_\mathfrak{m}$ (via the isomorphism $\wedge^* \mathbf{V}^* \cong \pi_* \mathcal{R}$ induced by the inverse of (15.7)) are compatible with one another, in the sense of (15.1).*

In view of the discussion of the paragraph following (15.12), this follows from the following Lemma.

Lemma 15.3. *Let notation be as above; in particular, Q_n and other data S_∞°, R_∞ etc. as in §14.1. The composite of the isomorphisms*

$$(15.15) \quad \pi_* \mathcal{R} \xrightarrow{(14.8)} \mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W) \xrightarrow{(15.11)} \wedge^*(t_S/t_R)^* \xrightarrow{(15.13)} \wedge^* V^*$$

coincides, in degree 1, with the map $\pi_1 \mathcal{R} \rightarrow V^$ constructed in (15.7).*

The map $\pi_1 \mathcal{R} \rightarrow V^*$ is constructed in a “natural” fashion, whereas the first map and the third map in sequence (15.15) depend on the various choices made to set up the situation of §14.1. Nonetheless the Lemma asserts that the composite is independent of all choices.

Proof. (of Lemma):

As above S_∞°, R_∞ are naturally augmented to W . For $n \geq m$ consider the following diagram, where Hom simply means homomorphisms of abelian groups:

$$(15.16) \quad \begin{array}{ccc} \mathrm{Hom}(\pi_1(R_\infty \underline{\otimes}_{S_\infty^\circ} W), W_m) & \longleftarrow & \pi_0 \left(\text{lifts to } R_\infty \underline{\otimes}_{S_\infty^\circ} W \rightarrow W_m \oplus W_m[1] \right) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(\pi_1(\overline{R}_n \underline{\otimes}_{S_n^\circ} W_n), W_m) & \longleftarrow & \pi_0 \left(\text{lifts to } \overline{R}_n \underline{\otimes}_{S_n^\circ} W_n \rightarrow W_m \oplus W_m[1] \right) \\ \downarrow (14.4) & & \downarrow \\ \mathrm{Hom}(\pi_1 \mathcal{R}, W_m) & \longleftarrow & \pi_0 \left(\text{lifts to } \mathcal{R} \rightarrow W_m \oplus W_m[1] \right). \end{array}$$

On the right hand side of this diagram, “lifts” always refers to lifting the natural map to the discrete ring W_m given by the augmentations; and the horizontal maps result by applying π_1 to such a lift.

Consider the middle row. Using (15.5), and the long exact sequence for André-Quillen cohomology of a derived tensor product¹⁴, we get a sequence:

$$\frac{D_W^0(\overline{S}_n^\circ, W_m)}{D_W^0(\overline{R}_n, W_m)} \rightarrow D_W^1(\overline{R}_n \underline{\otimes}_{S_n^\circ} W_n, W_m) \xrightarrow{\sim} \pi_0 \left(\text{lifts to } \overline{R}_n \underline{\otimes}_{S_n^\circ} W \rightarrow W_m \oplus W_m[1] \right).$$

Proceeding similarly for the top row, and using Lemma 15.1 for the bottom row, we arrive at the following diagram:

¹⁴This can be deduced from Lemma 4.30 (iv).

$$\begin{array}{ccc}
(15.17) & \text{Hom}(\pi_1(R_\infty \otimes_{S_\infty} W), W_m) & \xleftarrow{\sim} (\mathfrak{t}_S/\mathfrak{t}_R) \otimes W_m \\
& \uparrow f & \uparrow h \\
& \text{Hom}(\pi_1(\overline{R}_n \otimes_{\overline{S}_n} W_n), W_m) & \xleftarrow{\frac{D_W^0(\overline{S}_n^\circ, W_m)}{D_W^0(\overline{R}_n, W_m)}} \\
& \downarrow g & \downarrow \theta \\
& \text{Hom}(\pi_1 \mathcal{R}, W_m) & \xleftarrow[\text{Lemma 15.1}]{H_f^2(\mathbb{Z}[\frac{1}{S}], \text{Ad} \rho_m)}.
\end{array}$$

The top horizontal isomorphism is the dual of the isomorphism (15.9)

$$\underbrace{\text{Tor}_1^{S_\infty}(R_\infty, W)}_{=\pi_1(R_\infty \otimes_{S_\infty} W)} \cong (\mathfrak{t}_S/\mathfrak{t}_R)^*.$$

We need to identify the map $\theta : D_W^0(\overline{S}_n^\circ, W_m) \rightarrow H_f^2(\mathbb{Z}[\frac{1}{S}], \text{Ad} \rho_m)$ in the diagram above. This is of the deformation rings \mathcal{R}_S and $\mathcal{R}_{S_{Q_n}}$ – that is to say, the map β in (11.14); in our case:

$$(15.18) \quad \theta : \bigoplus_{v \in Q_n} \frac{H^1(\mathbb{Q}_v, \text{Ad} \rho_m)}{H_{\text{ur}}^1(\mathbb{Q}_v, \text{Ad} \rho_m)} \longrightarrow H_f^2(\mathbb{Z}[\frac{1}{S}], \text{Ad} \rho_m),$$

where the identification of $D_W^0(\overline{S}_n^\circ, W_m)$ with $\bigoplus_{v \in Q_n} \frac{H^1(\mathbb{Q}_v, \text{Ad} \rho_m)}{H_{\text{ur}}^1(\mathbb{Q}_v, \text{Ad} \rho_m)}$ is as in [37, §8.14].

We will now use the following fact, which can be proved by tracing through the definitions: for $\beta \in \bigoplus_{v \in Q_n} \frac{H^1(\mathbb{Q}_v, \text{Ad} \rho_m)}{H_{\text{ur}}^1(\mathbb{Q}_v, \text{Ad} \rho_m)}$ and $\alpha \in H_f^1(\mathbb{Z}[\frac{1}{S}], \text{Ad}^* \rho_m(1))$, the pairing $\langle \theta(\beta), \alpha \rangle$ of Tate global duality can be computed as the sum of local pairings $\sum_{v \in Q_n} \langle \beta_v, \alpha|_{\mathbb{Q}_v} \rangle$. This shows that θ coincides with the composite

$$(15.19) \quad \bigoplus_{v \in Q_n} \frac{H^1(\mathbb{Q}_v, \text{Ad} \rho_m)}{H_{\text{ur}}^1(\mathbb{Q}_v, \text{Ad} \rho_m)} \rightarrow H_f^1(\mathbb{Z}[\frac{1}{S}], \text{Ad}^* \rho_m(1))^* \rightarrow H_f^2(\mathbb{Z}[\frac{1}{S}], \text{Ad} \rho_m)$$

where the former map is induced by the sum of local pairings $\sum_{v \in Q_n} \langle \beta_v, \alpha|_{\mathbb{Q}_v} \rangle$ (this coincides with a map constructed in [37, §8.15]) and the latter map is induced by Tate global duality.¹⁵

Having identified θ , let us return to diagram (15.17). The middle row forms an inverse system over n (by means of the identifications of (b) of Theorem 13.1). We claim that all the maps in diagram (15.17) are compatible with this inverse system. For the maps f, h this is clear by definition; for the map g it is the homotopy compatibility discussed in §14.1; for the map θ it is discussed in [37, §8.25, after (162)].

¹⁵In the situation at hand, H_f^1 and so also H_f^2 is a free module over W_m , and the pairing of Tate global duality is perfect.

Therefore, we may obtain a new diagram by passing to an inverse limit over n . After one passes to the inverse limit over n the upper maps become isomorphisms; then we can invert the top layer of vertical maps in (15.17). The result is

$$(15.20) \quad \begin{array}{ccc} \mathrm{Hom}(\pi_1(R_\infty \otimes_{S_\infty} W), W_m) & \xleftarrow{(15.9)} & (t_S/t_R) \otimes W_m \\ \downarrow F & & \downarrow G \\ \mathrm{Hom}(\pi_1 \mathcal{R}, W_m) & \xleftarrow{\text{Lemma 15.1}} & H_f^2(\mathbb{Z}[\frac{1}{S}], \mathrm{Ad} \rho_m). \end{array}$$

where $F = g \circ (\varprojlim_n f)^{-1}$ and $G = \theta \circ (\varprojlim_n h)^{-1}$.

The isomorphism F is, by definition, exactly that of (14.8). Thus the composite $(t_S/t_R) \otimes W_m \rightarrow \mathrm{Hom}(\pi_1 \mathcal{R}, W_m)$ is the map induced by the first two arrows in (15.15). The equality of θ and the composite of (15.19) shows that the map G above coincides with the (reduction mod p^n of the) isomorphism induced by (15.13). (Unfortunately, to verify this requires wading into the maze of [37] to unravel the origin of (15.13): the relevant definitions, which match well with (15.19), are in [37, §8.15]). This concludes the proof of the Lemma, and so also of Theorem 15.2. \square

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APPENDIX A. HOMOTOPY COLIMITS AND HOMOTOPY LIMITS

We shall use the *Bousfield–Kan formula* for homotopy (co)limits of simplicial sets. We recall the definition and some well known properties here. Reference: Bousfield and Kan’s book, chapter XI and XII.

Notationwise, we shall write $C(x, y)$ for the set of morphisms from x to y in a category C . For *simplicially enriched categories* we shall also write $C(x, y)$ for the simplicial set of morphisms. For example, if X and Y are simplicial sets we shall write $s\text{Sets}(X, Y)$ for the simplicial set of maps $X \rightarrow Y$.

A.1. Nerves of categories. The set $[p] = \{0, \dots, p\}$ may be regarded as an ordered set using \leq , and hence a category, with one morphism $i \rightarrow j$ if $i \leq j$ and none otherwise. This gives a functor $\Delta \rightarrow \text{Cat}$.

If C is a small category, the *nerve* of C is the simplicial set whose p -simplices are the functors $[p] \rightarrow C$, i.e. p -tuples of composable morphisms. We write NC for the nerve and $N_p C$ for its set of p -simplices.

For an object $c \in C$, the *under category* $(c \downarrow C)$ has objects pairs (d, f) with $f : c \rightarrow d$ and morphisms commutative triangles. It comes with a forgetful functor $(c \downarrow C) \rightarrow C$ and we have a canonical functor

$$\begin{aligned} C^{\text{op}} &\rightarrow s\text{Sets} \\ c &\mapsto N(c \downarrow C). \end{aligned}$$

All values $N(c \downarrow C)$ are contractible simplicial sets, so the functor is naturally weakly equivalent to the terminal functor which takes all objects to a point.

A.2. Homotopy colimits. For a small category C and a simplicial set Y we obtain a functor

$$\begin{aligned} C &\rightarrow s\text{Sets} \\ c &\mapsto s\text{Sets}(N(c \downarrow C), Y), \end{aligned}$$

which we shall denote $s\text{Sets}(N(- \downarrow C), Y)$. The homotopy colimit of a functor $X : C \rightarrow s\text{Sets}$ is a simplicial set and has the universal property that the set of maps

$$f : \text{hocolim}_{c \in C} X(c) \rightarrow Y$$

are in natural bijection with the set of natural transformations

$$f : X \rightarrow s\text{Sets}(N(- \downarrow C), Y).$$

(And p -simplices in $s\text{Sets}(\text{hocolim} X, Y)$ are given by the same formula with Y replaced by $s\text{Sets}(\Delta[p], Y)$.) In other words, to specify a map $f : \text{hocolim} X \rightarrow Y$ amounts to specifying maps of simplicial sets $f_c : X(c) \times N(c \downarrow C) \rightarrow Y$ for all objects $c \in C$, in a way that is compatible with morphisms $c_0 \rightarrow c_1$ in C . A simplicial set $\text{hocolim} X$ with this universal property can be constructed as a

quotient of $\coprod_{c \in C} X(c) \times N(c \downarrow C)$: explicitly it is the coequalizer of a diagram

$$\coprod_{(c_0 \rightarrow c_1) \in N_1 C} X(c_0) \times N(c_1 \downarrow C) \rightrightarrows \coprod_{c \in N_0 C} X(c) \times N(c \downarrow C).$$

For example, if G is a group considered as a category with one element then a functor $G \rightarrow s\mathbf{Sets}$ is a simplicial set with an action and the homotopy colimit is just the “Borel construction”.

The most important property of homotopy colimits is their homotopy invariance, in the following sense.

Lemma A.1. *Let $F, G : C \rightarrow s\mathbf{Sets}$ be two functors, and let $T : F \rightarrow G$ be a natural transformation. If $T : F(c) \rightarrow G(c)$ is a weak equivalence for all objects c , then the induced map*

$$\mathrm{hocolim}_C F \xrightarrow{T} \mathrm{hocolim}_C G$$

is a weak equivalence. □

Recall that a category is *filtered* if for any objects c, c' of C there exists an object c'' and morphisms $c \rightarrow c'$ and $c \rightarrow c''$, and that any two parallel arrows $c \rightrightarrows c'$ become equal after composing with some arrow $c' \rightarrow c''$. For such categories the ordinary categorical colimit is already homotopy invariant.

Lemma A.2. *If C is filtered and $F : C \rightarrow s\mathbf{Sets}$ is a functor, then the natural map*

$$\mathrm{hocolim}_{c \in C} F(c) \rightarrow \mathrm{colim}_{c \in C} F(c)$$

is a weak equivalence. □

Despite this lemma, the homotopy colimit over a filtered category can still be quite useful: for example, it can be easier to define explicit maps out of the homotopy colimit.

Remark A.3. *In the special case $C = \mathbb{N}$ there is a convenient smaller model for the homotopy colimit of $X : \mathbb{N} \rightarrow s\mathbf{Sets}$. Namely for each $n \in \mathbb{N}$ we have the sub simplicial space*

$$\Delta[1] \cong N(n \downarrow \{n, n-1\}) \subset N(n \downarrow \mathbb{N}),$$

and the union of the sub simplicial sets $X(n) \times \Delta[1] \subset \mathrm{hocolim} X(n)$ is the mapping telescope of $X(0) \rightarrow X(1) \rightarrow \dots$, obtained by gluing the spaces $X(n) \times \Delta[1]$ along the maps $X(n) \rightarrow X(n+1)$. The inclusion of the telescope into the homotopy colimit is a weak equivalence.

A.3. Homotopy limits. The homotopy limit of a functor $Y : C^{\mathrm{op}} \rightarrow s\mathbf{Sets}$ is defined dually. The under category $(c \downarrow C^{\mathrm{op}})$ is opposite to the over category $(C \downarrow c)$, and gives a functor

$$\begin{aligned} C &\rightarrow s\mathbf{Sets} \\ c &\mapsto N(c \downarrow C^{\mathrm{op}}). \end{aligned}$$

The homotopy limit of a functor $Y : C^{\text{op}} \rightarrow s\text{Sets}$ is a simplicial set with the universal property that the set of maps

$$X \rightarrow \text{holim}_{c \in C^{\text{op}}} Y(c)$$

are in natural bijection with the set of natural transformations

$$X \times N(- \downarrow C^{\text{op}}) \rightarrow Y.$$

In fact, the homotopy limit can be regarded as the simplicial set of natural transformations $N(- \downarrow C^{\text{op}}) \Rightarrow Y$, where the simplicialness comes from the fact that both functors take values in the simplicially enriched category $s\text{Sets}$. Thus, $\text{holim} Y$ is the simplicial subspace

$$\text{holim} Y \subset \prod_{c \in N_0 C} s\text{Sets}(N(c \downarrow C^{\text{op}}), Y(c))$$

consisting of elements satisfying that for each $(C_0 \rightarrow C_1) \in N_1(C)$ the usual square (defining “naturality”) commutes, as a diagram of simplicial sets.

Example A.4. When C is the three-object category $C = (\bullet \leftarrow \bullet \rightarrow \bullet)$, a functor $Y : C^{\text{op}} \rightarrow s\text{Sets}$ is the same thing as a diagram of simplicial sets $Y_0 \rightarrow Y_{01} \leftarrow Y_1$. In this case the homotopy limit is often called the homotopy pullback, at least when all three spaces are Kan, and we shall denote it $Y_0 \times_{Y_{01}}^h Y_1$. Spelling out the definition, there is a bijection between maps $X \rightarrow Y_0 \times_{Y_{01}}^h Y_1$ and tuples $(f_0, f_1, f_{01}, h_0, h_1)$ consisting of maps

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ f_0 \downarrow & \searrow f_{01} & \downarrow g_1 \\ Y_0 & \xrightarrow{g_0} & Y_{01} \end{array}$$

and simplicial homotopies $h_0 : \Delta[1] \times X \rightarrow Y_{01}$ from $g_0 \circ f_0$ to f_{01} and $h_1 : \Delta[1] \times X \rightarrow Y_{01}$ from $g_1 \circ f_1$ to f_{01} .

We have the following nice adjunction formula

Lemma A.5. Let $X : C \rightarrow s\text{Sets}$ be a functor and Y be a simplicial set. Then we have a natural isomorphism of simplicial sets

$$s\text{Sets}(\text{hocolim}_{c \in C} X(c), Y) \cong \text{holim}_{c \in C^{\text{op}}} s\text{Sets}(X(c), Y).$$

Lemma A.6. Let $Y \rightarrow Y'$ be a natural transformation of functors $C^{\text{op}} \rightarrow s\text{Sets}$ be a functor such that $Y(c) \rightarrow Y'(c)$ is a weak equivalence of Kan simplicial sets for all objects $c \in C$. Then the induced map $\text{holim} Y \rightarrow \text{holim} Y'$ is a weak equivalence.

However, we caution the reader that the Bousfield–Kan formula for the homotopy limit of $Y \rightarrow s\text{Sets}$ does *not* have good homotopical properties unless all values $Y(c)$ are Kan. In fact it might not even agree with what other authors (such as Quillen) call “homotopy limit” in such cases.

A.3.1. *Products.* The following lemma follows from the definition.

Lemma A.7. *Let $X, Y : C \rightarrow s\mathbf{Sets}$ be two functors and let $X \times Y : C \rightarrow s\mathbf{Sets}$ denote the objectwise cartesian product. Then the natural map*

$$\mathrm{holim}(X \times Y) \rightarrow (\mathrm{holim}X) \times (\mathrm{holim}Y)$$

is an isomorphism of simplicial sets. \square

Corollary A.8. *Let \mathbf{SCR} denote the category of simplicial commutative rings and let $X : C \rightarrow \mathbf{SCR}$ be a diagram of such. Then $\mathrm{holim}X$ is naturally a simplicial commutative ring again. Similarly for simplicial modules, simplicial abelian groups, etc.* \square

\mathbf{SCR} has the structure of a *simplicial model category*, and hence it makes sense to talk about “internal” homotopy limits and colimits of functors $C \rightarrow \mathbf{SCR}$. The above corollary shows that the forgetful functor $\mathbf{SCR} \rightarrow s\mathbf{Sets}$ commutes with homotopy limits, just as the forgetful map from commutative rings to sets commutes with ordinary limits.

A.4. Calculation of homotopy limits and colimits. Let us briefly discuss a few tools for manipulating homotopy limits and colimits.

A.4.1. *Cofinality.* If D is a category and $X : D \rightarrow s\mathbf{Sets}$ and $Y : D^{\mathrm{op}} \rightarrow s\mathbf{Sets}$ are diagrams, then any functor $F : C \rightarrow D$ induce maps of simplicial sets

$$\begin{aligned} \mathrm{hocolim}_C(X \circ F) &\rightarrow \mathrm{hocolim}_D X \\ \mathrm{holim}_{D^{\mathrm{op}}} Y &\rightarrow \mathrm{holim}_{C^{\mathrm{op}}} Y \circ F. \end{aligned}$$

These maps are both weak equivalences, provided F is *cofinal* in the strong sense that for all objects $d \in D$ the under category $(d \downarrow F)$, whose objects are pairs (c, f) with $c \in C$ and $f : d \rightarrow F(c)$, have *contractible* nerve. (Assuming only that these under categories have *connected* nerves is sufficient for the corresponding maps for limits and colimits of sets to be bijections.)

Let us also recall from [SGA 4, Expose 1, 8.1.6] or [Edwards–Hastings 2.1.6] that any filtered category D admits a cofinal functor $t : C \rightarrow D$ with C a *directed set*. The standard proof is to let C be the set of finite subdiagrams $A \subset D$ which has a unique final element. Then C is ordered by inclusion and $t : C \rightarrow D$ sends a diagram to its final element. Let us point out that the resulting directed set C comes with an order preserving map $C \rightarrow \mathbb{N}$ given by sending A to its cardinality. Thus, when calculating homotopy limits or colimits over filtered categories, we may assume that the indexing category is a directed set equipped with an order preserving map to \mathbb{N} .

A.4.2. *Homotopy groups of homotopy limits.* It can be difficult to understand the homotopy groups of a homotopy limit, even when the indexing category is filtered and even for π_0 . Under an additional assumption on either the indexing category or on the values of the functor we can say more.

Lemma A.9. *If C is countable and filtered and $F : C^{\text{op}} \rightarrow s\text{Sets}$ has Kan values, then*

$$\pi_0 \text{holim} F(c) \rightarrow \lim \pi_0 F(c)$$

is surjective. For any $k \geq 0$ and any point $x \in \text{holim} F(c)$ there is a short exact sequence of groups (pointed sets for $k = 0$)

$$\lim_{c \in C^{\text{op}}} {}^1\pi_{k+1}(F(c), x) \rightarrow \pi_k(\text{holim}_{c \in C^{\text{op}}} F(c), x) \rightarrow \lim_{c \in C^{\text{op}}} \pi_k(F(c), x).$$

Proof sketch. The lemma is clear if C has a final element. If this is not the case, countability of C implies that there is a cofinal functor $\mathbb{N} \rightarrow C$, so without loss of generality we may just assume $C = \mathbb{N}$. An element in $\lim \pi_0 F(n)$ may then be represented by zero-simplices $x_n \in F(n)$ such that there exists 1-simplices $h_n : \Delta[1] \rightarrow F(n)$ from x_n to the image of x_{n+1} . Then use the Kan-ness of the $F(n)$ to inductively extend the h_n to compatible maps $N(n \downarrow \mathbb{N}^{\text{op}}) \rightarrow F(n)$. (Or alternatively use a different model for the homotopy limit, dual to the telescope model for homotopy colimit, in which no further data than the (x_n, h_n) is required.) \square

Without the countability assumption on C it can apparently happen that for example $\pi_0 \text{holim} F \rightarrow \lim \pi_0 F$ is not surjective. However, if we assume that the values of F have finite homotopy groups we may use Tychonoff's theorem to rule out this kind of behavior.

Proposition A.10. *Let C be filtered and $F : C^{\text{op}} \rightarrow s\text{Sets}$ be a functor whose values are Kan and have $\pi_0(F(c))$ finite for all c and $\pi_k(F(c), x)$ finite for all k and all $x \in F(c)$. Then the natural map*

$$\pi_0 \text{holim} F(c) \rightarrow \lim \pi_0 F(c)$$

is a bijection and, for any point $x \in \text{holim} F(c)$, so is the natural map

$$\pi_k(\text{holim} F(c), x) \rightarrow \lim \pi_k(F(c), x).$$

Proof sketch. It should be well known that this follows from the Bousfield–Kan spectral sequence and the vanishing of higher derived limits of finite groups over filtered categories. Let us outline a “manual” argument.

It suffices to prove the claim about π_0 , since holim commutes with based loop spaces. We shall outline the argument for surjectivity of the map $\pi_0 \text{holim} \rightarrow \lim \pi_0$. An element in $\lim \pi_0 F(c)$ determines a sub-functor of F with path connected values, so it suffice to prove that if $F(c)$ is path connected for all c then $\text{holim} F(c)$ is non-empty.

Without loss of generality C is a directed set such that $\{c \in C \mid c \leq d\}$ is finite for all d and hence that any finite collection of objects is contained in a finite sub poset which has a maximal terminal element. Let $x_c \in F(c)$ be a vertex, and choose for all pairs $c_0 < c_1$ a 1-simplex $h_{c_0 < c_1} : \Delta[1] \rightarrow F(c_0)$ from x_{c_0} to the image of $x_{c_1} \in F(c_1) \rightarrow F(c_0)$. These 1-simplices may be assembled to define compatible maps $N_{\leq 1}(c \downarrow C^{\text{op}}) \rightarrow F(c)$, where $N_{\leq 1}$ denotes the 1-skeleton of the nerve. The problem is that these maps may not admit extensions over the 2-skeleton, and

in fact there is an obstruction in $\pi_1(F(c_0), x_{c_0})$ for each $c_0 < c_1 < c_2$, given by concatenating the (images in $F(c_0)$ of) the paths $h_{c_0 < c_1}$, $h_{c_0 < c_2}$, and $h_{c_1 < c_2}$ in the appropriate order. These obstructions assemble to an element of

$$(A.1) \quad \prod_{c_0 < c_1 < c_2} \pi_1(F(c_0), x_{c_0})$$

which vanishes if and only if the maps $N_{\leq 1}(c \downarrow C^{\text{op}}) \rightarrow F(c)$ extend to compatible maps over the two-skeleton. We decide to keep the x_c and rechoose the $h_{c_0 < c_1}$. The homotopy class of $h_{c_0 < c_1}$ relative to its endpoints is a torsor for $\pi_1(F(c_0), x_{c_0})$, and acting simultaneously for all $(c_0 < c_1)$ gives a map

$$(A.2) \quad \prod_{c_0 < c_1} \pi_1(F(c_0), x_{c_0}) \rightarrow \prod_{c_0 < c_1 < c_2} \pi_1(F(c_0), x_{c_0}).$$

If we can show that this map is surjective, the paths $h_{c_0 < c_1}$ may be rechosen to make the obstruction element in (A.1) vanish. Such a re-choice is always possible on a sub poset of the form $\{c \in C \mid c \leq d\}$ so the image of (A.2) surjects onto any finite product of the factors. Another way to say this is that the image of (A.2) is dense in the product topology on $\prod_{c_0 < c_1 < c_2} \pi_1(F(c_0), x_{c_0})$ when each $\pi_1(F(c_0), x_{c_0})$ is given the discrete topology. If each $\pi_1(F(c_0), x_{c_0})$ is finite, the source of (A.2) is compact and hence its image is too so density of the image implies surjectivity, and hence there is no obstruction to rechoosing the $h_{c_0 < c_1}$ and get compatible extensions to maps

$$N_{\leq 2}(c \downarrow C^{\text{op}}) \rightarrow F(c).$$

Extending this map over the 3-skeletons we encounter obstructions in the cokernel of a homomorphism

$$\prod_{c_0 < c_1 < c_2} \pi_2(F(c_0), x_{c_0}) \rightarrow \prod_{c_0 < c_1 < c_2 < c_3} \pi_2(F(c_0), x_{c_0}),$$

which is surjective by a similar argument (in fact, its cokernel is precisely the derived limit $\lim^3 \pi_2(F(c), x_c)$). Continuing this way we end up with maps $N(c \downarrow C^{\text{op}}) \rightarrow F(c)$, compatible over varying $c \in C$, and hence a point in $\text{holim} F$. Injectivity is similar, with obstructions vanishing for the same reason as $\lim^k \pi_k(F(c))$ vanishes. \square

As the proof shows, one can sometimes get by with slightly weaker assumptions in the above Proposition, e.g. that $\pi_k(F(c))$ is a compact group/set for some topology in which the functoriality is continuous.

Corollary A.11. *Let C be a filtered category, $F : C \rightarrow s\text{Sets}$ a functor, and Z is a Kan simplicial set such that $[F(c), Z] = \pi_0 s\text{Sets}(F(c), Z)$ is finite for all c and $\pi_k(s\text{Sets}(F(c), Z), f)$ is finite for all k and all $c \in C$ and all $f : F(c) \rightarrow Z$. (This happens e.g. if Z has finite homotopy groups and $F(c)$ is equivalent to a finite CW complex for all c .) Then the natural map*

$$[\text{hocolim}_{c \in C} F(c), Z] \rightarrow \lim_{c \in C} [F(c), Z]$$

is a bijection. In other words, any collection of maps $F(c) \rightarrow Z$ which are compatible up to some (unspecified) homotopies may in fact be glued together to a map out of the homotopy colimit, and uniquely so up to homotopy. \square

Corollary A.12. *Let X be a simplicial set and $F : C^{\text{op}} \rightarrow s\text{Sets}$ be a functor with Kan values, such that $s\text{Sets}(X, F(c))$ has finite homotopy groups for all $c \in C$. Then the natural map*

$$[X, \text{holim}_{c \in C} F(c)] \rightarrow \lim_{c \in C} [X, F(c)]$$

is a bijection.

We shall not directly use these two results, but the analogous results in the similar setting of functors into simplicial commutative rings will be very useful.

APPENDIX B. DUALITY AND LOCAL CONDITIONS IN GALOIS COHOMOLOGY

We formulate a version of Poitou–Tate duality with local constraints, following a suggestion of Harris to work with cone constructions. This is implicit in [7] and surely known to all experts.

B.1. Statement of the theorem.

B.2. We work in étale cohomology of $\mathbb{Z}[\frac{1}{S}]$ where $p \in S$. Let M be a p -torsion étale locally constant sheaf, which we may think of as a representation of the S -unramified quotient $\text{Gal}(\mathbb{Q}_S/\mathbb{Q})$ of the Galois group; we write simply $H^i(M)$ for the étale cohomology or $H^i(\mathbb{Z}_S, M)$ where the set S must be made explicit.

We fix once and for all algebraic closures $\overline{\mathbb{Q}_v}$ and embeddings $\iota_v : \mathbb{Q}_S \hookrightarrow \overline{\mathbb{Q}_v}$ for each $v \in S$. This induces a map on Galois groups

$$\iota_v^* : \text{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \rightarrow \text{Gal}(\mathbb{Q}_S/\mathbb{Q}).$$

We write $H^*(\mathbb{Q}_v, M)$ for the Galois cohomology of $H^*(\text{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), M)$.

If G is any group and M a G -module, we refer to the “standard cochain complex” of *inhomogeneous* cochains computing the cohomology $H^*(G, M)$; for example, a 2-cochain is a function $G \times G \rightarrow M$, etc. Note that G acts on this complex by conjugation on the domain (e.g. $G \times G$ in the example just given), this action descends to the trivial action on cohomology. Also the cup product lifts to the cochain level in the standard (back face, front face) way. If G is a profinite group and M a discrete G -module we will always understand cochains to be continuous.

Let $C^*(\mathbb{Q}_v, M)$ be the standard cochain complex computing Galois cohomology of $H^*(\text{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), M)$, and similarly define $C^*(M)$ as the standard cochain complex computing Galois cohomology $H^*(\text{Gal}(\mathbb{Q}_S/\mathbb{Q}), M)$. For brevity, if the module M is understood, we will sometimes refer (e.g.) to $C^2(M)$ as C^2 and $C^2(\mathbb{Q}_v, M)$ as C_v^2 .

If x is a cochain in the standard cochain complex computing the cohomology of $\text{Gal}(\mathbb{Q}_S/\mathbb{Q})$ we use the notation $x|_{\mathbb{Q}_v}$ – or simply x_v if there is no risk of confusion

– for the cochain obtained by pulling back x under the map $\iota_v^* : \text{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \rightarrow \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$.

B.3. Local conditions, lifted to the cochain level. We want to impose “local conditions” $\mathcal{L}_v^i \subset H^i(\mathbb{Q}_v, M)$ at places in S ; we will usually just write for short $\mathcal{L}_v \subset H^*(\mathbb{Q}_v, M)$. There will be a long exact sequence

$$H_{\mathcal{L}}^*(M) \rightarrow H^*(M) \rightarrow \prod_{v \in S} H^*(\mathbb{Q}_v, M)/\mathcal{L} \xrightarrow{[1]}$$

where $H_{\mathcal{L}}^*$ are “cohomology classes that belong to \mathcal{L} .”

To achieve this precisely, we need lifts to the co-chain level. More precisely, we will suppose that \mathcal{L}_v comes equipped with a subcomplex $C_{\mathcal{L},v}^* \subset C^*(\mathbb{Q}_v, M)$ satisfying the following axioms:

- (i) $C_{\mathcal{L},v}^*$ is closed under the differential, and
- (ii) $C_{\mathcal{L},v}^*$ is invariant under conjugacy, and
- (iii) The cohomology of $C^*(\mathbb{Q}_v, M)/C_{\mathcal{L}}^*(\mathbb{Q}_v, M)$ “is” H^*/\mathcal{L} , i.e. the natural map from the cohomology of $C^*(\mathbb{Q}_v, M)$ to the cohomology of $C^*(\mathbb{Q}_v, M)/C_{\mathcal{L}}^*(\mathbb{Q}_v, M)$ is surjective in each degree and its kernel is precisely \mathcal{L}_v .

We define $H_{\mathcal{L}}^i$ to be the *derived* set of classes in $H^i(M)$ that lie inside \mathcal{L} for each $v \in S$, that is to say the cohomology of the cone

$$(B.1) \quad C^n(M) \oplus \bigoplus_{v \in S} \frac{C^{n-1}(\mathbb{Q}_v, M)}{C_{\mathcal{L}}^{n-1}(\mathbb{Q}_v, M)}$$

We denote an element of this group by (x, y_v) , where y_v really denotes an element of $C^{n-1}/C_{\mathcal{L}}^{n-1}$ for each $v \in S$. The differential is the “cone” differential, that is to say, $d(x, y_v) = (-dx, dy_v + x|_v)$.

To be explicit: cocycle in this group is a pair

$$(x \in C^n(M), y_v \in C^{n-1}(\mathbb{Q}_v, M)/C_{\mathcal{L}}^{n-1}(\mathbb{Q}_v, M))$$

with the property that $x|_{\mathbb{Q}_v} + d(y_v)$ belongs to $C_{\mathcal{L}}^n$, i.e. “ x equipped with a reason for its restriction to \mathbb{Q}_v to belong to \mathcal{L} .”

Write $M = \text{Hom}(M^*, \mu_{p^\infty})$. In what follows, we will want to consider also a dual local condition \mathcal{L}^\perp . By this, we mean that we take the orthogonal complement $\mathcal{L}^\perp \subset H^*(\mathbb{Q}_v, M^*)$ and *assume* that it is equipped with a similar enrichment to the cochain level, $C_{\mathcal{L}^\perp} \subset C(\mathbb{Q}_v, M)$, which satisfies (i)–(iii) above and additionally

- (iv) the cup product

$$(B.2) \quad C_{\mathcal{L}}^i(\mathbb{Q}_v, M) \times C_{\mathcal{L}^\perp}^{3-i}(\mathbb{Q}_v, M^*) \rightarrow C^3(\mathbb{Q}_v, \mu_{p^\infty})$$

vanishes *at the chain level*.

We are ready to formulate the statement of duality:

Theorem B.1. *Suppose $p > 2$, that $\mathcal{L}, \mathcal{L}^\perp$ are as above, and both equipped with lifts to the cochain level, where both lifts satisfy (i)–(iv) above. There is a duality*

$$H_{\mathcal{L}}^i(M) \times H_{\mathcal{L}^\perp}^{3-i}(M^*) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

where as usual $M^* = \text{Hom}(M, \mu_{p^\infty})$.

B.4. Examples of local conditions meeting our conditions. The most important example, besides the trivial example¹⁶ is the following generalization of “unramified” local conditions:

Take an arbitrary subgroup $\mathfrak{l} \subset H^1$, and take

$$\mathcal{L} = \begin{cases} H^0 \subset H^0 \\ \mathfrak{l} \subset H^1 \\ 0 \subset H^2 \end{cases}, \quad \mathcal{L}^\perp = \begin{cases} H^0 \subset H^0 \\ \mathfrak{l}^\perp \subset H^1 \\ 0 \subset H^2 \end{cases}.$$

We take, both for \mathcal{L} (respectively \mathcal{L}^\perp):

- $C_{\mathcal{L}}^0 = C_0$.
- $C_{\mathcal{L}}^1$ to consist of all classes $x \in C^1(\mathbb{Q}_v, M)$ where $dx = 0$ and $[x] \in \mathfrak{l}$ (resp. \mathfrak{l}^\perp)
- $C_{\mathcal{L}}^j = 0$ for $j \geq 2$.

The complex $C/C_{\mathcal{L}}$ looks like

$$0 \rightarrow C^1(\mathbb{Q}_v, M)/C_{\mathcal{L}}^1 \rightarrow C^2(\mathbb{Q}_v, M) \rightarrow \dots$$

and visibly the cohomology groups are 0, $H^1/\mathcal{L}_1, H^2$ as desired. The vanishing of the desired cup product (B.2) is obvious.

B.5. Proof of the theorem.

B.5.1. First we verify this for $i = 0$. Given

$$\alpha \in \ker(H^0(M) \rightarrow \prod_{v \in S} H^0(\mathbb{Q}_v, M)/\mathcal{L}_v^0)$$

$$\beta \in \text{coker} \left(H^2(M^*) \rightarrow \prod_{v \in S} H^2(\mathbb{Q}_v, M^*)/\mathcal{L}_{\perp, v}^2 \right)$$

we define $\langle \alpha, \beta \rangle$ as the sum of local reciprocity pairings

$$\sum_{v \in S} (\alpha_v, \beta_v).$$

This is well-defined because in fact $\alpha|_{\mathbb{Q}_v} \in \mathcal{L}_v^0$ for each $v \in S$. We claim it is a perfect pairing.

¹⁶ take $\mathcal{L} = 0$ in all degrees, with $C_{\mathcal{L}}(\mathbb{Q}_v, M) = 0$ also; dually take $\mathcal{L}^\perp = H^i$ in all degrees, with $C_{\mathcal{L}^\perp}(\mathbb{Q}_v, M^*) = C(\mathbb{Q}_v, M^*)$

The perfection of this pairing amounts to the statement that the resulting map

$$(B.3) \quad \frac{\prod_{v \in S} H^2(\mathbb{Q}_v, M^*)}{\langle \mathcal{L}_{\perp, v}^2, H^2(M^*) \rangle} \rightarrow \ker(H^0(M) \rightarrow \prod_{v \in S} H^0(\mathbb{Q}_v, M)/\mathcal{L}_v)^\vee$$

is an isomorphism. Here $(\dots)^\vee$ on the right hand side means maps to \mathbb{Q}/\mathbb{Z} ; in what follows, let us use the word “functional” for “map to \mathbb{Q}/\mathbb{Z} .”

The map (B.3) is visibly surjective: the nine-term exact sequence asserts that $\prod_{v \in S} H^2(\mathbb{Q}_v, M^*) \rightarrow H^0(M)^\vee$ is surjective (which is obvious anyway). For injectivity, suppose that $\beta \in \prod_{v \in S} H^2(\mathbb{Q}_v, M^*)$ induces the zero functional on the right-hand side. In particular “pairing with β ” descends to a functional on the image of $H^0(M)$ inside $\prod_v H^0(\mathbb{Q}_v, M)/\mathcal{L}_v^0$. If we replace β by $\beta + s_v$ for $s_v \in \mathcal{L}_{\perp, v}^2$, the induced functional on this image changes the restriction of $\langle -, s_v \rangle$ on $\prod_v H^0(\mathbb{Q}_v, M)/\mathcal{L}_v$. But the pairing

$$\mathcal{L}_{2, v}^\perp \times H^0(\mathbb{Q}_v, M)/\mathcal{L}_v^0 \rightarrow \mathbb{Q}/\mathbb{Z}$$

is an isomorphism, and so we can modify β by an element of $\prod \mathcal{L}_{\perp, v}^2$ in such a way that the functional “pairing with β ” is actually zero on the image of $H^0(M)$. Then the nine-term exact sequence means that β actually lies in the image of $H^2(M^*)$, as desired.

B.5.2. Now we examine the trickier case $i = 1$. The main issue is to construct the pairing. We then verify it is perfect by filtering the groups involved and looking at the pairing on graded pieces, where it reduces to better-known pairings.

Take classes

$$(x \in C^1, \overline{y_v} \in C_v^0/C_{\mathcal{L}, v}^0)$$

and

$$(x' \in C^2, \overline{y'_v} \in C_v^1/C_{\mathcal{L}^\perp, v}^1)$$

representing elements of $H_{\mathcal{L}}^1$ and $H_{\mathcal{L}^\perp}^2$. Lift $\overline{y_v}, \overline{y'_v}$ to $y_v \in C_{\mathcal{L}, v}^0, y'_v \in C_{\mathcal{L}^\perp, v}^1$. Set

$$\epsilon_v = dy_v + x_v \in C_{v, \mathcal{L}}^1, \quad \epsilon'_v = dy'_v + x'_v \in C_{\mathcal{L}^\perp, v}^2$$

similarly. Note that $d\epsilon_v = 0$. Take $z \in C^2$ with $dz = x \cup x'$. Now form

$$(B.4) \quad (y_v \cup x'_v) - (\epsilon_v \cup y'_v) + z_v \in C^2(\mathbb{Q}_v, \mu_{p^\infty})$$

The differential of this equals

$$\epsilon_v \cup x'_v + \epsilon_v \cup dy'_v = (\epsilon_v \cup \epsilon'_v) \stackrel{(iv)}{=} 0.$$

where we used assumption (iv).

In other words, we have a class in $H^2(\mathbb{Q}_v, \mu_{p^\infty})$ for each $v \in S$. Taking the sum of invariants - evidently independent of choice of z - gives an element of $\mathbb{Q}_p/\mathbb{Z}_p$ associated to the classes (x, y_v) and (x', y'_v) . This is our pairing

$$(B.5) \quad H_{\mathcal{L}}^1(M) \times H_{\mathcal{L}^\perp}^2(M^*) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

B.5.3. Symmetry. This construction is “symmetric,” or rather a similar construction with roles reversed gives the same: if we consider, instead of (B.4), the class

$$(B.6) \quad -x_v \cup y'_v + y_v \cup \epsilon'_v + z_v$$

and form an invariant similarly, the result is the same.

In fact, the differential of (B.6) is $x_v \cup \epsilon'_v + dy_v \cup \epsilon'_v = \epsilon_v \cup \epsilon'_v = 0$; and (B.4) differs from (B.6) by

$$y_v \cup x'_v + x_v \cup y'_v - \epsilon_v \cup y'_v - y_v \cup \epsilon'_v = -y_v \cup dy'_v - dy_v \cup y'_v = -d(y_v \cup y'_v)$$

thus they have the same cohomology class.

B.5.4. Independence. Let's try to see this is independent of the various choices we made.

- (a) If we modify y_v by $w_v \in C^0_{v,\mathcal{L}}$ the class (B.4) changes by

$$w_v \cup x'_v - dw_v \cup y'_v = w_v \cup (x'_v + dy'_v) - \underbrace{d(w_v \cup y'_v)}_{dw_v \cup y'_v + w_v \cup dy'_v}$$

which is cohomologically trivial.

If we modify y'_v by w'_v the situation is similar.

- (b) Suppose we modify x by a boundary in the fashion ($x \mapsto x_v - da, y_v \mapsto y_v + a$), for some $a \in C^0$. This does not change ϵ_v . Replacing z with $z - a \cup x'$, we see that the class (B.4) is unchanged.

B.5.5. Perfect pairing. We now verify that (B.5) is perfect, by comparing it to standard pairings. We have filtrations on $H^i_{\mathcal{L}}$ with successive graded pieces as follows:

$$\underbrace{\text{cokernel}(H^{i-1} \rightarrow \prod_v H^{i-1}(\mathbb{Q}_v)/\mathcal{L})}_{x_v = \epsilon_v = 0}, \underbrace{\text{Sha}^i}_{\epsilon_v = 0}, \text{image}(H^i \rightarrow \prod_v H^i(\mathbb{Q}_v)/\mathcal{L}),$$

e.g. the first piece comes from the image in cohomology of cycles satisfying $x_v = \epsilon_v = 0$ for all $v \in S$. As usual, we define Sha^i here to be global cohomology classes that are everywhere locally trivial.

It's easy to see that this filtration is its own dual under this pairing. We will explicate the pairing piece by piece, with respect to this filtration:

$$(B.7) \quad \text{cokernel}(H^0 \rightarrow \prod_v H^0(\mathbb{Q}_v, M)/\mathcal{L}_v^0) \times \text{image}(H^2 \rightarrow \prod_v H^2(M^*)/\mathcal{L}_{\perp,v}^2)$$

$$(B.8) \quad \text{Sha}^1 \times \text{Sha}^2$$

$$(B.9) \quad \text{image}(H^1 \rightarrow \prod_v H^1(M^*)/\mathcal{L}_{v,\perp}^1) \times \text{cokernel}(H^1 \rightarrow \prod_v H^1(M)/\mathcal{L}_v^1)$$

For the first: we take $(x = 0, y_v \in H^0(\mathbb{Q}_v))$ representing the left-hand class, and on the right we can take any pair (x', y'_v) with x' representing the given class in the image. We can take $z = 0$. Then (B.4) is given by

$$y_v \cup x'_v - dy_v \cup y'_v \sim y_v \cup (x'_v - dy'_v)$$

and thus realizes the standard local duality of H^0 and H^2 .

For (B.8): it's easy to see the pairing from above recovers the Tate pairing.

For (B.9): We can choose a representative for the right-hand class with $x' = \epsilon'_v = 0$; on the left we choose a representative (x_v, y_v) . Again we can take $z = 0$. Then (B.4) is given by $-\epsilon_v \cup y'_v = -(x_v + dy_v) \cup y'_v$, which realizes up to sign the standard local pairing of H^1 and H^1 .

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